Computing the Isotropic Ratio Landscape on Polygons

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a senior thesis submitted in partial fulfillment
of the requirements for the degree of
Bachelor of Arts in Mathematics at
Princeton University

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April 26, 2026

Abstract

Mathematicians have conjectured that the simplexes are the polytopes that *maximize* the isotropic ratio among all convex bodies in $\mathbb{R}^n$ (the Strong Slicing Conjecture). This has been established only in $\mathbb{R}^2$, where the triangle is the unique maximizer. In this paper we use both theoretical and computational approaches to derive explicit formulas for the first and second derivatives of the isotropic ratio of a polytope in $\mathbb{R}^2$ in terms of its vertices, with the aim of providing a foundation for attacking the conjecture in higher dimensions.

Acknowledgements

Folks in the math department: I would like to thank my advisor Professor Ramon Van Handel for advising my thesis this year and my second reader Professor Zeev Dvir. I would also like to thanks Shouda Wang for helping me understand the framework behind perturbations and for teaching me Chinese too. I'd like to also give a special shoutout to David Villalobos for helping tutor me from MAT 215/217 to running a mock thesis defense session with me. Finally, I would also like to thank Dr. Alice Chang for her advising my junior year on the Isoperimetric Inequality which sparked my interest in convex geometry.

Outside the math department: I'm thankful to God and Princeton for the resources I had these past four years to study something that truly interests me. I'd like to thank my close friends and family for always encouraging me, sending food to me, or even visiting me a month before graduation. Finally, I'd like to thank Manna, Princeton Christian Fellowship, The Scully Coop, the Chinese Department and the Princeton Swim Club for filling my past four years with so much fun and excitement!

Declaration

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Chapter 1

Introduction

In the study of geometry, understanding the characteristics of convex bodies and polytopes is of central importance. A classical result in this direction is the Isoperimetric Inequality in $\mathbb{R}^2$: among all simple closed curves of fixed length, the circle encloses the largest area. Various mathematicians contributed to this inequality such as Jakob Steiner, who showed that any maximizer must be a circle (without establishing existence of a maximizer), and by Karl Weierstrass, who completed the argument using the calculus of variations. *The Isoperimetric Inequality in* $\mathbb{R}^2$ states, for any simple closed curve γ in $\mathbb{R}^2$ with length $L(\gamma)$ and enclosed area $A(\gamma)$,:

$$\frac{L(\gamma)^2}{A(\gamma)} \geq 4\pi,$$

with equality if and only if γ is a circle.

One might ask about candidates that *maximize* $L(\gamma)^2/A(\gamma)$. No maximizer exists: if γ is a rectangle with fixed perimeter, stretching it along its longer side decreases $A(\gamma)$ without bound, so $L(\gamma)^2/A(\gamma) \to \infty$ along a sequence whose limit is a degenerate line segment. The set of curves with fixed perimeter is therefore not compact, and no maximizer is attained.

Instead, we turn our attention to a related quantity that measures the *shape* of a convex body more intrinsically: the isotropic ratio. Unlike the isoperimetric problem — which fixes $L(\gamma)$ and asks which shapes maximize $A(\gamma)$ — the isotropic problem asks: among all convex bodies with a prescribed covariance structure $C(P) = I_n$ and barycenter at the origin $b_P = 0$ (isotropic position), which ones maximize volume?

Unlike the isoperimetric case, the isotropic ratio *does* have both a maximizer and a minimizer among all convex bodies in $\mathbb{R}^n$. As we will later see, since the isotropic ratio is invariant under affine transformations (Lemma 2.1.2), theoretically it suffices to restrict our attention to bodies already in isotropic position. Specifically, by Lemma 2.2.1, every convex body K can be placed in isotropic position via the map $T(x) = C(K)^{-1/2}(x - b_K)$.

The Euclidean ball B_2^n is known to be the *minimizer* of the isotropic ratio among all convex bodies in $\mathbb{R}^n$ [9]. The *maximizers* are conjectured to be the simplexes (the Strong Slicing Conjecture). The Strong Slicing Conjecture suggests that for any convex body $K \subseteq \mathbb{R}^n$,

$$L_K \leq L_{\triangle^n} = \frac{(n!)^{\frac{1}{n}}}{(n+1)^{\frac{n+1}{2n}} \cdot \sqrt{n+2}}$$

where $\triangle^n \subseteq \mathbb{R}^n$ is any simplex whose vertices span $\mathbb{R}^n$ and sum to zero and L_K^2 is the isotropic ratio (Definition 2.1.7). This has been confirmed in $\mathbb{R}^2$, where the triangle is the unique maximizer [13, 14]. The Weak Slicing Conjecture — that L_K^2 is bounded above by a universal constant c independent of dimension — was recently resolved [2, 10, 11], but the Strong Slicing Conjecture remains open.

Unfortunately, solving the Strong Slicing Conjecture will not cure cancer, however proving this conjecture would prove other open conjectures that involve affine-invariant geometric quantities such as Mahler's Conjecture [7, 15], so developing new tools to study them is interesting. In this paper we aim to provide a computational foundation for attacking the Strong Slicing Conjecture by using explicit formulas for the first and second derivatives of L_K^2 in $\mathbb{R}^2$ on polygons to gather more information about the landscape of the isotropic ratio on polygons.

As with most optimization problems, the natural approach is to compute the gradient ∇L_K^2 and Hessian $\nabla^2 L_K^2$, find critical points where $\nabla L_K^2 = 0$, and then use the sign of $\nabla^2 L_K^2$ to classify them as local maxima or minima. However, first-order conditions alone are insufficient: for instance, both the isotropic square and the isotropic triangle in $\mathbb{R}^2$ satisfy first-order conditions, yet a direct computation shows the triangle has a strictly larger isotropic ratio [3]. Computing these derivatives by hand for $n \geq 3$ on a single polytope is difficult, so we hope to run computed experiments to help us formulate conjectures. Below are a few questions we are interested in:

1. What happens at points which are not local minimizers of the isotropic ratio?
2. Is there a bound on the number of non-negative eigenvalues in the Hessian for non-local minimizers?
3. Can we conjecture about the general behavior for non-minimizers and minimizers?

Our strategy is:

Compute the first and second derivatives of the isotropic ratio of a polytope P in $\mathbb{R}^2$ (polygon) in terms of its vertices, and see if we can find any interesting patterns in the Hessian/gradient that we might generalize to higher dimensions.

We begin by establishing the necessary definitions and background results.

Chapter 2

Background

2.1 Definitions

Remark 2.1.1 (Standing assumption). Throughout this paper, all polytopes are assumed to be convex, compact with non-empty interior, with the origin in the interior. Perturbing the vertices v_i by a sufficiently small amount preserves these properties.

Definition 2.1.1. Polytopes (H-Representation)

A **polytope** P is defined as the intersection of m half-spaces (assumed bounded):

$$P = \{x \in \mathbb{R}^n : \langle n_i, x \rangle \leq 1 \text{ for all } i \leq m\}$$

where the n_i are outer normal vectors to the facets F_i of P.

A **facet** of P is $F_i = \{x \in P : \langle n_i, x \rangle = 1\}$, and the boundary of P is $\partial P = \bigcup_{i=1}^{m} F_i$.

Definition 2.1.2. Polytopes (V-Representation)

Let $v_1, .., v_k \in \mathbb{R}^n$, $k \in \mathbb{N}$ finite, $conv(v_1, v_2, ., v_k)$ is the convex hull of the vectors $v_1, v_2, ., v_k$. then a **polytope** $P = conv(v_1, v_2, .., v_k)$ is a polytope. (Assume origin is in the interior)

A vertex v_i of a polytope P can be represented as $v_i = (v_{i1}, v_{i2}, .., v_{in})$. In $\mathbb{R}^2, \mathbb{R}^3$ we may use notation like v_{ix} or d_{iy} to denote the x/y component of v_i. In general n case, we will use v_{ij}.

Definition 2.1.3. Affine Transformation

An affine transformation of $\mathbb{R}^n$ is a map $f : \mathbb{R}^n \to \mathbb{R}^n$ where $f(p) = Ap+b, \forall p \in \mathbb{R}^n, A \in GL_n(\mathbb{R})$

Definition 2.1.4. Barycenter of P (First Moment of P)

$$b_P = \frac{1}{\mathrm{Vol}(P)} \int_P x dx, x \in P$$

Definition 2.1.5. Covariance Matrix of P

The covariance matrix of P is

$$C(P) \ = \ \frac{1}{\mathrm{Vol}(P)} \int_P (x - b_P)(x - b_P)^T \, dx.$$

Equivalently, writing $x \sim \mathrm{Uniform}(P)$ and $b_P = \mathbb{E}[x]$, this is $C(P) = \mathrm{Cov}(x)$. An alternative closed form is:

$$C(P) = \frac{\int_P xx^T \, dx}{\mathrm{Vol}(P)} - \frac{\int_P x \, dx}{\mathrm{Vol}(P)} \cdot \frac{\int_P x^T \, dx}{\mathrm{Vol}(P)}.$$

Proof of equivalence. Let $x \sim \mathrm{Uniform}(P)$. Then

$$\begin{aligned} C(P) &= \mathbb{E}[(x - b_P)(x - b_P)^T] \\ &= \mathbb{E}[xx^T] - \mathbb{E}[x]b_P^T - b_P\mathbb{E}[x^T] + b_P b_P^T \\ &= \mathbb{E}[xx^T] - b_P b_P^T, \end{aligned}$$

which is precisely $\frac{\int_P xx^T \, dx}{\mathrm{Vol}(P)} - \frac{\int_P x \, dx}{\mathrm{Vol}(P)} \cdot \frac{\int_P x^T \, dx}{\mathrm{Vol}(P)}$. ☐

Definition 2.1.6. Isotropic Position

A convex body P is said to be in isotropic position if the following conditions are met

1. $b_P = 0$

2. $C(P) = I_n$ (identity matrix)

Definition 2.1.7. Isotropic Ratio

The **isotropic ratio** of a convex body P is

$$L_P^2 = \left(\frac{\det C(P)}{\mathrm{Vol}(P)^2} \right)^{1/n}.$$

Definition 2.1.8. Simplices in $\mathbb{R}^n$

A simplex S_n in $\mathbb{R}^n$ with nonempty interior is defined to be the convex hull of $n + 1$ affinely independent vectors.

Lemma 2.1.2. *Given any convex body P, $L_P^2 = L_{TP}^2$ where $Tx = Ax + b$, $A \in GL_n(\mathbb{R})$, $b \in \mathbb{R}^n$ (the isotropic ratio is invariant under affine transformations).*

Proof. We compute $\mathrm{Vol}(TP)$, b_{TP}, and $C(TP)$ individually, using $C(P) = \frac{1}{\mathrm{Vol}(P)} \int_P (x - b_P)(x - b_P)^T \, dx$ (Definition 2.5).

1. $\mathrm{Vol}(TP) = |\det A| \,\mathrm{Vol}(P)$, by the change-of-variables formula for linear maps.

2. $b_{TP} = \frac{1}{\mathrm{Vol}(TP)} \int_{TP} p \, dp$. Substituting $p = Ax + b$ and applying the change of variables $dp = |\det A| \, dx$:

$$b_{TP} = \frac{|\det A|}{|\det A| \,\mathrm{Vol}(P)} \int_P (Ax + b) \, dx = \frac{1}{\mathrm{Vol}(P)} \left(A \int_P x \, dx + b \,\mathrm{Vol}(P) \right) = Ab_P + b.$$

3. $C(TP) = \frac{1}{\mathrm{Vol}(TP)} \int_{TP} (p - b_{TP})(p - b_{TP})^T \, dp$. Substituting $p = Ax + b$ (so $p - b_{TP} = A(x - b_P)$) and $dp = |\det A| \, dx$:

$$C(TP) = \frac{|\det A|}{|\det A| \,\mathrm{Vol}(P)} \int_P A(x - b_P)(x - b_P)^T A^T \, dx = AC(P)A^T.$$

Substituting into the isotropic ratio:

$$L_{TP}^2 = \left(\frac{\det C(TP)}{\mathrm{Vol}(TP)^2} \right)^{1/n} = \left(\frac{\det(AC(P)A^T)}{|\det A|^2 \,\mathrm{Vol}(P)^2} \right)^{1/n}$$

$$= \left(\frac{|\det A|^2 \det C(P)}{|\det A|^2 \,\mathrm{Vol}(P)^2} \right)^{1/n} = \left(\frac{\det C(P)}{\mathrm{Vol}(P)^2} \right)^{1/n} = L_P^2. \qquad \square$$

2.2 Existence of Isotropic Position

Lemma 2.1 shows that the isotropic ratio is an affine invariant. A natural question is: does every convex body actually *admit* an isotropic position, i.e. can we always find an affine map T such that $T(K)$ satisfies $b_{T(K)} = 0$ and $C(T(K)) = I_n$? The answer is yes, and the canonical choice is $T(x) = C(K)^{-1/2}(x - b_K)$.

Lemma 2.2.1. *Let $K \subset \mathbb{R}^n$ be a convex body with barycenter b_K and (positive-definite) covariance matrix $C(K)$. Define the affine map*

$$T(x) = M\,(x - b_K), \qquad M = C(K)^{-1/2},$$

where $C(K)^{-1/2}$ denotes the (unique, symmetric, positive-definite) matrix square root of $C(K)^{-1}$. Then $T(K)$ is in isotropic position, i.e.

$$b_{T(K)} = 0 \quad \textit{and} \quad C(T(K)) = I_n.$$

Proof. We verify the two isotropic conditions separately.

Step 1: Barycenter. Since $T(x) = M(x - b_K)$ is an affine map with linear part $A = M$ and translation $b = -Mb_K$, we have from the proof of Lemma 2.1:

$$b_{T(K)} = A\, b_K + b = M\, b_K + (-M\, b_K) = 0. \qquad \checkmark$$

Step 2: Covariance. From Lemma 2.1 (item 3 in the proof), $C(T(K)) = A\, C(K)\, A^T$. With $A = M = C(K)^{-1/2}$ and $A^T = M^T = C(K)^{-1/2}$ (since $C(K)^{-1/2}$ is symmetric):

$$C(T(K)) = C(K)^{-1/2}\, C(K)\, C(K)^{-1/2}.$$

By definition of the matrix square root, $C(K)^{-1/2}\, C(K)^{1/2} = I_n$, so

$$C(K)^{-1/2}\, C(K)\, C(K)^{-1/2} = \big(C(K)^{-1/2}\, C(K)^{1/2}\big)\big(C(K)^{1/2}\, C(K)^{-1/2}\big) = I_n \cdot I_n = I_n. \ \checkmark$$

Therefore $T(K)$ satisfies both conditions of Definition 2.6 and is in isotropic position. $\square$

Remark 2.2.2. The choice $M = C(K)^{-1/2}$ is not unique: any map of the form $M = U\, C(K)^{-1/2}$ with U orthogonal also places K in isotropic position (since $U\, I_n\, U^T = I_n$). The symmetric choice $M = C(K)^{-1/2}$ is canonical because it is the unique positive-definite square root.

Furthermore, the resulting isotropic body $T(K)$ has the same isotropic ratio as K by Lemma 2.1. This justifies the remark in the Introduction: without loss of generality, one may always assume that the convex body under study is already in isotropic position when analyzing the isotropic ratio.

Now that we have defined our definitions, we can start deriving the first and second derivatives of L_P^2. The first step is to define a perturbation which L_P^2 will be changing with respect to. Whenever one talks about optimizing functions over polytopes, we should think of observing how this function behaves over this polytope under some perturbation. We can define this perturbation rigorously by assigning a perturbation vector w_i to every outer normal vector, n_i, of each facet F_i.

Definition 2.2.1. Perturbation of P

Let $w_i \in \mathbb{R}^n, \forall i, \|w_i\|= 1$. Define $W =$

$$\begin{bmatrix} -w_1- \\ -w_2- \\ ... \\ -w_m- \end{bmatrix}$$

which is the array of rows $(w_0, w_1, ..w_m)$. Let $P_{t,W}$ be P perturbed by W with respect to time $t, 0 \leq t \leq 1$.

We define $P_{t,W} = \{x \in \mathbb{R} : \langle n_i + tw_i, x\rangle \leq 1\}$ $\forall i, F_{i,w_i,t} = \{x \in P_{t,W} : \langle n_i + tw_i, x\rangle = 1\}$ denotes the ith-facet of $P_{t,W}$.

Definition 2.2.2. Velocity of ∂P

For all $x \in \partial P$, draw a line in the direction of its outer normal in P. We denote $d_t(x)$ to be the signed distance between x and the point along this line where it hits $\partial P_{t,W}$ then we can define the velocity

$$v(x) = \lim_{t\to 0} \frac{d_t(x)}{t}$$

Lemma 2.2.3. *Let $f : \mathbb{R}^n \to \mathbb{R}, P_{t,W}$ and ∂P denote our perturbed polytope and its boundary at $t = 0$. Then,*

$$\frac{d}{dt} \int_{P_{t,W}} f(x)dx \Bigg|_{t=0} = \int_{\partial P} v(x)f(x)dx$$

Proof. By the Leibniz Integral Rule, we have $\frac{d}{dt} \int_{P_{t,W}} f(x,t)dx \Big|_{t=0} = \int_{P_{t,W}} (\frac{\partial}{\partial t} f(x,t))dx + \int_{\partial P}(f(x,t)\mathbf{v} \cdot \mathbf{n})dx$ where $\mathbf{n}$ is the outer unit normal vector at x and $\mathbf{v}$ is the velocity vector field. Since our f does not depend on t, $\int_{P_{t,W}} \frac{\partial}{\partial t} f(x,t)dx = 0$, and one can check that $\mathbf{v} \cdot \mathbf{n}$ is exactly defined by $v(x)$ in Definition 3.2. $\square$

Lemma 2.2.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and*

$$H(n_1, .., n_m) = \int_P f(x)dx$$

where $n_1, .., n_m$ are the outer normal vectors that define P then

$$\nabla_i H(n_1, .., n_m) = -\int_{F_i} \frac{x}{\|n_i\|} f(x) dx$$

Proof. Notice, we already have that $\frac{d}{dt}\int_{P_{t,W}} f(x)dx\Big|_{t=0} = \int_{\partial P} v(x)f(x)dx$, so it suffices to find a representation of $v(x)$ with respect to a perturbation that only adjusts F_i (any perturbation that adjusts facets $F_j, j \neq i$ does not change v_i and hence $\nabla_i H(v_1, .., v_m) = 0$). We can represent these perturbation as the following: for some $i, 1 \leq i \leq m, n_i = n_i + tw_i$ for some $w_i \in \mathbb{R}^n$ (i.e. only one facet is shifted by w). It follows that

$$\langle n_i + tw_i, x + d_t(x)\frac{n_i}{\|n_i\|}\rangle = 1$$

since $n_i + tw_i$ is a point that lies on $F_{i,w,t}$ by definition, and x is exactly $d_t(x)$ away from its intersection with $F_{i,w,t}$ in the $\frac{n_i}{\|n_i\|}$ direction. Hence, the two points live on the same facet $F_{i,w,t}$.

$$\langle n_i + tw_i, x + d_t(x)\frac{n_i}{\|n_i\|}\rangle = \langle n_i + tw_i, x\rangle + \langle n_i + tw_i, d_t(x)\frac{n_i}{\|n_i\|}\rangle$$

$$= \langle n_i, x\rangle + \langle tw_i, x\rangle + \langle n_i, d_t(x)\frac{n_i}{\|n_i\|}\rangle + \langle tw_i, d_t(x)\frac{n_i}{\|n_i\|}\rangle$$

Note, $\langle n_i, x\rangle = 1$ so

$$\langle tw_i, x\rangle + \langle n_i, d_t(x)\frac{n_i}{\|n_i\|}\rangle + \langle tw_i, d_t(x)\frac{n_i}{\|n_i\|}\rangle = 0$$

$$\Rightarrow v(x) = -\frac{\langle w_i, x\rangle}{\|n_i\|}$$

$$\langle w_i, \nabla_i H\rangle = \frac{d}{dt}\int_{P_{t,W}} f(x)dx\Bigg|_{t=0} = \int_{\partial P} v(x)f(x)dx = -\int_{F_{i,w,t}} \frac{\langle w_i, x\rangle}{\|n_i\|} f(x)dx$$

□

Lemma 2.2.5. *To find maximizers P of*

$$L_k^2 = (\frac{\det C(P)}{Vol(P)^2})^{\frac{1}{n}}$$

it suffices to find minimizers P of

$$\overline{L_k^2} = \frac{Vol(P)^{2n+2}}{\det \overline{C}(P)}$$

where $\overline{C}(P) = Vol(P)\int_P xx^T dx - \int_P xdx \int_P x^T dx$

Proof. Note, $\overline{C}(P) = \mathrm{Vol}(P)^2 C(P) \Rightarrow \det C(P)\mathrm{Vol}(P)^{2n} = \det \overline{C}(P)$

$$\overline{L_P^2} = \frac{\mathrm{Vol}(P)^{2n+2}}{\det \overline{\mathrm{C}}(P)} = \frac{\mathrm{Vol}(P)^{2n+2}}{\det(\mathrm{Vol}(P)^2 C(P))} = \frac{\mathrm{Vol}(P)^{2n+2}}{\mathrm{Vol}(P)^{2n}\det C(P)} = \frac{\mathrm{Vol}(P)^2}{\det C(P)} = (L_P^2)^{-n}$$

Hence,

$$(L_P^2)^{-n} = \frac{1}{(L_P^2)^n} = \overline{L_P^2}$$

Since $(L_P^2)^n$ is a monotonic increasing function, maximizers of L_P^2 must be minimizers of $\overline{L_P^2}$.

□

Chapter 3

Deriving the First and Second Derivative of the Isotropic Ratio

As mentioned earlier, one of the classic optimization approaches to finding the maximizers/minimizers is to find where the first derivative of our function is 0, and use the second derivative to rule out candidates. As shown in Lemma 3.2, it suffices to find minimizers of $\frac{\mathrm{Vol}(P_{t,W})^{2n+2}}{\det \bar{C}(P_{t,W})}$ with respect to time and some set of perturbation vectors W. Let our function

$$f_W(t) = \frac{\mathrm{Vol}(P_{t,W})^{2n+2}}{\det \bar{C}(P_{t,W})}$$

below we compute $\frac{d}{dt} f_W(t)$ and $\frac{d^2}{dt^2} f_W(t)$.

Lemma 3.0.1.

$$\frac{d}{dt} f_W(t) = \frac{Vol(P_{t,W})^{2n+1}}{\det \bar{C}(P_{t,W})} \left[(2n+2) \frac{d}{dt} Vol(P_{t,W}) - Vol(P_{t,W})\, Tr\left[\bar{C}(P_{t,W})^{-1} \frac{d}{dt} \bar{C}(P_{t,W}) \right] \right]$$

Proof. By Quotient Rule,

$$\frac{d}{dt} \frac{\mathrm{Vol}(P_{t,W})^{2n+2}}{\det \bar{C}(P_{t,W})} = \frac{\det \overline{C}(P_{t,W})(2n+2)\mathrm{Vol}(P_{t,W})^{2n+1} \frac{d}{dt}\mathrm{Vol}(P_{t,W}) - \mathrm{Vol}(P_{t,W})^{2n+2} \frac{d}{dt}(\det \bar{C}(P_{t,W}))}{\det \bar{C}(P_{t,W})^2}$$

By Jacobi's Formula ($\frac{d}{dt} \det A(t) = \det A(t)[\mathrm{Tr}(A(t)^{-1} \frac{d}{dt} A(t)]$

$$= \frac{\det \overline{C}(P_{t,W})(2n+2)\mathrm{Vol}(P_{t,W})^{2n+1} \frac{d}{dt}\mathrm{Vol}(P_{t,W}) - \mathrm{Vol}(P_{t,W})^{2n+2}[\det \bar{C}(P_{t,W})\mathrm{Tr}[\overline{C}(P_{t,W})^{-1} \frac{d}{dt}\overline{C}(P_{t,W})]]}{\det \bar{C}(P_{t,W})^2}$$

After some rearrangement of terms,

$$= \frac{\mathrm{Vol}(P_{t,W})^{2n+1}}{\det \overline{C}(P_{t,W})} \left[(2n+2)\frac{d}{dt}\mathrm{Vol}(P_{t,W}) - \mathrm{Vol}(P_{t,W})\mathrm{Tr}[\overline{C}(P_{t,W})^{-1}\frac{d}{dt}\overline{C}(P_{t,W})] \right]$$

$\square$

Lemma 3.0.2.

$$\frac{d^2}{dt^2}f_W(t) = \frac{V^{2n}}{D}\left[((2n+1)\frac{d}{dt}V - g)((2n+2)\frac{d}{dt}V - g) + (2n+2)V\frac{d^2}{dt^2}V - V^2\frac{d}{dt}\frac{g}{V} - g\frac{d}{dt}V) \right]$$

Proof. Let $g_W(t) = \mathrm{Vol}(P_{t,W})\mathrm{Tr}[\overline{C}(P_{t,W})^{-1}\frac{d}{dt}\overline{C}(P_{t,W})]$.

Then, $\frac{d}{dt}f_W(t) = \frac{\mathrm{Vol}(P_{t,W})^{2n+1}}{\det \overline{C}(P_{t,W})}\left[(2n+2)\frac{d}{dt}\mathrm{Vol}(P_{t,W}) - g_W(t)\right]$. To further simplify our calculations, we will denote $V = \mathrm{Vol}(P_{t,W})$ and $D = \det \overline{C}(P_{t,W})$. Then,

$$\frac{d}{dt}f_W(t) = \frac{V^{2n+1}}{D}\left[(2n+2)\frac{d}{dt}V - g_W(t)\right]$$

Let $A(t) = \frac{V^{2n+1}}{D}$ and $B(t) = (2n+2)\frac{d}{dt}V - g_W(t)$.

By Product Rule,

$$\frac{d^2}{dt^2}f_W(t) = [\frac{d}{dt}A(t)]B(t) + [\frac{d}{dt}B(t)]A(t)$$

Next, we compute $\frac{d}{dt}A(t)$ and $\frac{d}{dt}B(t)$

$$\frac{d}{dt}A(t) = \frac{(2n+1)V^{2n}\frac{d}{dt}VD - V^{2n+1}\frac{d}{dt}D}{D^2}$$

$$\frac{d}{dt}B(t) = (2n+2)\frac{d^2}{dt^2}V - \frac{d}{dt}g_W(t)$$

We substitute these expressions back into $\frac{d^2}{dt^2}f_W(t)$,

$$\frac{d^2}{dt^2}f_W(t) = [\frac{(2n+1)V^{2n}\frac{d}{dt}VD - V^{2n+1}\frac{d}{dt}D}{D^2}]B(t) + [(2n+2)\frac{d^2}{dt^2}V - \frac{d}{dt}g_W(t)]A(t)$$

$$= \frac{V^{2n}}{D}\left[((2n+1)\frac{d}{dt}V - \frac{V\frac{d}{dt}D}{D})((2n+2)\frac{d}{dt}V - g) + ((2n+2)\frac{d^2}{dt^2}V - \frac{d}{dt}g)V \right]$$

Recall, by Jacobi's Formula,

$$\frac{V\frac{d}{dt}D}{D} = \frac{V(D\mathrm{Tr}(\overline{C}(P_{t,W})^{-1}\frac{d}{dt}\overline{C}(P_{t,W})))}{D} = V * \mathrm{Tr}(\overline{C}(P_{t,W})^{-1}\frac{d}{dt}\overline{C}(P_{t,W}))) = g$$

If we substitute this back into $\frac{d^2}{dt^2}f_W(t)$, we get

$$\frac{d^2}{dt^2}f_W(t) = \frac{V^{2n}}{D}\left[((2n+1)\frac{d}{dt}V - g)((2n+2)\frac{d}{dt}V - g) + ((2n+2)\frac{d^2}{dt^2}V - \frac{d}{dt}g)V\right]$$

Notice, $V^2\frac{d}{dt}[\frac{g}{V}] + g\frac{d}{dt}V = V\frac{d}{dt}g$. Hence,

$$\frac{d^2}{dt^2}f_W(t) = \frac{V^{2n}}{D}\left[((2n+1)\frac{d}{dt}V - g)((2n+2)\frac{d}{dt}V - g) + (2n+2)V\frac{d^2}{dt^2}V - V^2\frac{d}{dt}\frac{g}{V} - g\frac{d}{dt}V)\right]$$

□

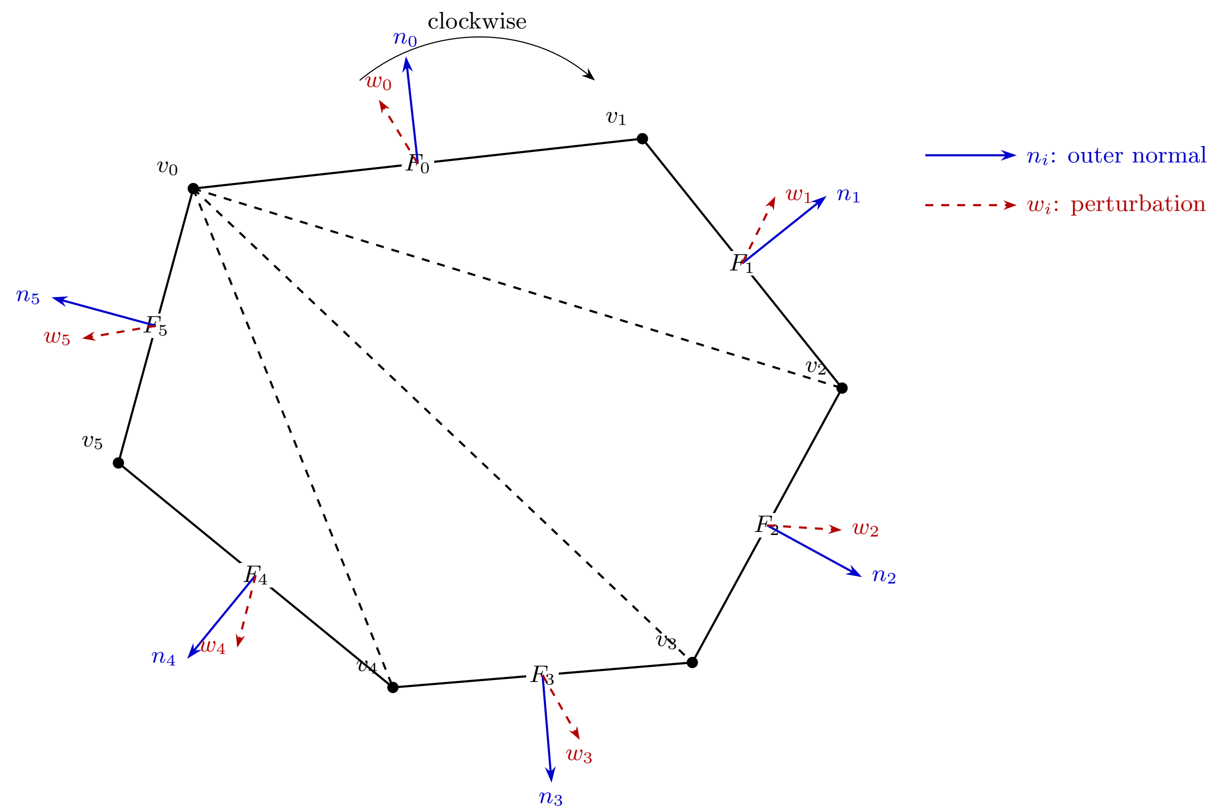
Chapter 4

Polygon Triangulation

In order to compute the first and second derivative of the isotropic ratio, we need expressions to compute the components $\text{Vol}(P_{t,W})$, $\bar{C}(P_{t,W})$, $\frac{d}{dt}\text{Vol}(P_{t,W})$, $\frac{d}{dt}\bar{C}(P_{t,W})$ in $\mathbb{R}^2$ in terms of the vertices of P. To do this, we start off by defining a triangulation of our polygon P, as these equations are easier to compute on triangles.

Let P be a polygon with m vertices. Let the set of vertices $V(P) = \{v_0, v_1, .., v_{m-1}\}$ where $\forall i, 0 \leq i \leq m-1, v_i = (v_{ix}, v_{iy})$ and $v_{ix}, v_{iy} \in \mathbb{R}$. We will denote, $v_i = v_{i \mod m}, \forall i \in \mathbb{N}$.

Label the vertices of P in clockwise orientation. Next, we triangulate P as follows:
$\forall i, 1 \leq i \leq m-2, \triangle_i = \triangle v_0 v_i v_{i+1}$



The **triangulation of P** in $\mathbb{R}^2$ is then defined to be the set of $m-2$ triangles $\triangle_i$ indexed by $1 \le i \le m-2$.

Chapter 5

The Moment Generating Function (MGF)

Now that we have defined our triangulation of a polygon P. Let us return to computing the first and second derivatives of the isotropic ratio as mentioned in Lemma 3.0.1, 3.0.2. One can compute these quantities directly by solving double integrals over the set of triangulations. However, we can also utilize the moment generating function to create a systematic template to computing these quantities over standard triangles. However, to illustrate the efficiency of moment generating functions let us first attempt to compute $\mathrm{Vol}(P_{t,W})$, $\int_{P_{t,W}} pp^T \, dp$ directly.

5.1 Computing $\mathrm{Vol}(P_{t,W})$

In general,

$$\mathrm{Vol}(P_{t,W}) = \int_{P_{t,W}} dp, \text{ where } p = (x, y) \in P_{t,W} \text{ and } x, y \in \mathbb{R}$$

In the case of $\mathbb{R}^2$, we can use our triangulation to compute $Vol(P)$.

$$\mathrm{Vol}(P_{t,W}) = \sum_{i=1}^{m-2} Area(\triangle_i)$$

where

$$Area(\triangle_i) = \frac{1}{2} |v_{ix} v_{(i+1)y} - v_{iy} v_{(i+1)x}|$$

Proof. Let $E_i = \begin{bmatrix} -v_i- \\ -v_{i+1}- \end{bmatrix}$. Then, $\det E_i = v_{ix}v_{(i+1)y} - v_{iy}v_{(i+1)x}$. Recall that $|\det E_i|$ represents the area of the parallelogram spanned by v_i and v_{i+1} (the absolute value takes care of the negative sign introduced from the vertices' clockwise orientation). Hence, $\text{Area}(\triangle_i) = \frac{1}{2}|\det E_i|$. We conclude with $\text{Vol}(P_{t,W}) = \sum_{i=1}^{m-2} \text{Area}(\triangle_i)$. □

5.2 Computing $\overline{C}(P_{t,W})$

Recall p is some point in $P_{t,W}$ with the representation $p = xe_1 + ye_2$ where $x, y \in \mathbb{R}$, $e_1 = (1,0)$, $e_2 = (0,1)$.

$$\bar{C}(P_{t,W}) = \text{Vol}(P_{t,W}) \int_{P_{t,W}} pp^T \, dp - \int_{P_{t,W}} pdp \int_{P_{t,W}} p^T \, dp$$

Hence, to compute $\overline{C}(P_{t,W})$, it boils down to computing $\int_{\triangle_i} pdp$, $\int_{\triangle_i} p^T dp$, $\int_{\triangle_i} pp^T dp$. Unlike in the volume case where we didn't have to evaluate an integral, here we do. In fact, each of these integrals are over triangles with different vertices so the computation gets messier. This is where the moment generating function comes in.

5.3 Defining our Moment Generating Function

Recall for a scalar random variable X, $M_X(\lambda) = \mathbb{E}[e^{\lambda t}] = \sum_{i=0}^{\infty} \frac{t^i \mathbb{E}[X^t]}{i!}$. Hence, one can take $\frac{d^n}{dt^n} M_X(0)$ to get $\mathbb{E}[X^n]$.

For a vector valued random variable Y, $M_Y(z) = \mathbb{E}[e^{\langle z,p \rangle}], z \in \mathbb{R}^2$.

In our case, we define $M_P(\lambda z) = \int_{P_{t,W}} e^{\lambda \langle z,p \rangle} dp = \sum_{i=1}^{m-2} \int_{\triangle_i} e^{\lambda \langle z,p \rangle} dp, z \in \mathbb{R}^2$

Theorem 5.3.1. *The* ***first form*** *of the Moment Generating Function (MGF) of a polygon P with m sides under the triangulation mentioned in Chapter 4 is given by*

$$M_P(\lambda z) = \sum_{i=1}^{m-2} \frac{|\det A_i|}{\lambda^2(a_i - c)} \left[\frac{e^{\lambda b_i} - e^{\lambda a_i}}{b_i - a_i} - \frac{e^{\lambda b_i} - e^{\lambda c}}{b_i - c} \right]$$

where $a_i = \langle z, v_i \rangle, b_i = \langle z, v_{i+1} \rangle, c = \langle z, v_0 \rangle, \lambda \in \mathbb{R}, z \in \mathbb{R}^2$

Proof. We will first derive the MGF on a special type of polygon with $v_0 = 0$ and then use this to derive the general form of the MGF where v_0 is not the origin.

5.3.1 Deriving the MGF on P with $v_0 = 0$

Let $M_{\triangle_i}(\lambda z) = \int_{\triangle_i} e^{\lambda\langle z,p\rangle} dp \Rightarrow M_P(\lambda z) = \sum_{i=1}^{m-2} M_{\triangle_i}(\lambda z)$. In order to compute $M_{\triangle_i}(\lambda z)$, we map $\triangle_i$ onto an easier domain to work with: the standard triangle $S\triangle = \triangle(\mathbf{0}, e_1, e_2)$ where e_i are the standard basis vectors in $\mathbb{R}^n$. This mapping is given by

$$\overline{A_i} = \begin{bmatrix} | & | \\ v_{i+1} & v_i \\ | & | \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

such that $A_i e_1 = v_{i+1}$, $A_i e_2 = v_i$ (assuming $v_0 = 0$).

For all $p \in \triangle_i$ we have $p = xe_1 + ye_2$ where $x, y \in \mathbb{R}$. Now, we compute the individual $M_{\triangle_i}(\lambda z)$,

$$M_{\triangle_i}(\lambda z) = \int_{\triangle_i} e^{\lambda\langle z,p\rangle}\, dp = |\det \overline{A_i}| \int_{S\triangle} e^{\lambda\langle z,\overline{A_i}p\rangle}\, dp = |\det \overline{A_i}| \int_{S\triangle} e^{\lambda(\langle z,\overline{A_i}e_1\rangle x + \langle z,\overline{A_i}e_2\rangle y)}\, dp$$

Noting $\overline{A_i}e_1 = v_{i+1}, \overline{A_i}e_2 = v_i$,

$$= |\det \overline{A_i}| \int_{S\triangle} e^{\lambda(\langle z,v_{i+1}\rangle x + \langle z,v_i\rangle y)} dp = |\det \overline{A_i}| \int_0^1 e^{\lambda\langle z,v_{i+1}\rangle x} \int_0^{1-x} e^{\lambda\langle z,v_i\rangle y}\, dy\, dx$$

$$= |\det \overline{A_i}| \int_0^1 e^{\lambda\langle z,v_{i+1}\rangle x} \left[\frac{e^{\lambda\langle z,v_i\rangle(1-x)} - 1}{\lambda\langle z,v_i\rangle}\right] dx$$

Factoring out $\frac{1}{\lambda\langle z,v_i\rangle}$,

$$= \frac{|\det \overline{A_i}|}{\lambda\langle z,v_i\rangle} \int_0^1 e^{\lambda\langle z,v_{i+1}\rangle x + \lambda\langle z,v_i\rangle(1-x)} - e^{\lambda\langle z,v_{i+1}\rangle}\, dx$$

$$= \frac{|\det \overline{A_i}|}{\lambda\langle z,v_i\rangle} \left[e^{\lambda\langle z,v_i\rangle} \int_0^1 e^{\lambda\langle z,v_{i+1}-v_i\rangle x}\, dx - \int_0^1 e^{\lambda\langle z,v_{i+1}\rangle x}\, dx\right]$$

Evaluating the integrals,

$$= \frac{|\det \overline{A_i}|}{\lambda\langle z,v_i\rangle} \left[e^{\lambda\langle z,v_i\rangle} \left[\frac{e^{\lambda\langle z,v_{i+1}-v_i\rangle} - 1}{\lambda\langle z,v_{i+1}-v_i\rangle}\right] - \left[\frac{e^{\lambda\langle z,v_{i+1}\rangle} - 1}{\lambda\langle z,v_{i+1}\rangle}\right]\right]$$

$$= \frac{|\det \overline{A_i}|}{\lambda^2 \langle z, v_i \rangle} \left[\frac{e^{\lambda \langle z, v_{i+1} \rangle} - e^{\lambda \langle z, v_i \rangle}}{\langle z, v_{i+1} - v_i \rangle} - \frac{e^{\lambda \langle z, v_{i+1} \rangle} - 1}{\langle z, v_{i+1} \rangle} \right]$$

Letting $a_i = \langle z, v_i \rangle, b_i = \langle z, v_{i+1} \rangle$,

$$= \frac{|\det \overline{A_i}|}{\lambda^2 a_i} \left[\frac{e^{\lambda b_i} - e^{\lambda a_i}}{b_i - a_i} - \frac{e^{\lambda b_i} - 1}{b_i} \right]$$

Hence,

$$M_P(\lambda z) = \sum_{i=1}^{m-2} \frac{|\det \overline{A_i}|}{\lambda^2 a_i} \left[\frac{e^{\lambda b_i} - e^{\lambda a_i}}{b_i - a_i} - \frac{e^{\lambda b_i} - 1}{b_i} \right]$$

5.3.2 Deriving the MGF on P with $v_0 \neq 0$

Notice if P does not have $v_0 = 0$ then we can simply translate $P \to P - v_0$ such that $v_0 = 0$, then apply the MGF we just derived above on the domain $P - v_0$. In order to account for this translation, we must add back v_0 to p within the integral. It suffices, to just replace v_i, v_{i+1} with $v_i - v_0$, $v_{i+1} - v_0$ respectively in $M_p(\lambda z)$ and $\overline{A_i}$ becomes A_i, where

$$A_i = \begin{bmatrix} | & | \\ v_{i+1} - v_0 & v_i - v_0 \\ | & | \end{bmatrix} \in \mathbb{R}^{2 \times 2}$$

More formally,

$$\int_P e^{\lambda \langle z, p \rangle} \, dp = \int_{P - v_0} e^{\lambda \langle z, p + v_0 \rangle} \, dp = e^{\lambda \langle z, v_0 \rangle} \int_{P - v_0} e^{\lambda \langle z, p \rangle} \, dp = e^{\lambda \langle z, v_0 \rangle} M_{P - v_0}(\lambda z)$$

$$= e^{\lambda \langle z, v_0 \rangle} \sum_{i=1}^{m-2} \frac{|\det A_i|}{\lambda^2 \langle z, v_i - v_0 \rangle} \left[\frac{e^{\lambda \langle z, v_{i+1} - v_0 \rangle} - e^{\lambda \langle z, v_i - v_0 \rangle}}{\langle z, (v_{i+1} - v_0) - (v_i - v_0) \rangle} - \frac{e^{\lambda \langle z, v_{i+1} - v_0 \rangle} - 1}{\langle z, v_{i+1} - v_0 \rangle} \right]$$

$$= e^{\lambda \langle z, v_0 \rangle} \sum_{i=1}^{m-2} \frac{|\det A_i|}{\lambda^2 \langle z, v_i - v_0 \rangle} \left[\frac{e^{\lambda \langle z, v_{i+1} - v_0 \rangle} - e^{\lambda \langle z, v_i - v_0 \rangle}}{\langle z, v_{i+1} - v_i \rangle} - \frac{e^{\lambda \langle z, v_{i+1} - v_0 \rangle} - 1}{\langle z, v_{i+1} - v_0 \rangle} \right]$$

$$= \sum_{i=1}^{m-2} \frac{|\det A_i|}{\lambda^2 (a_i - c)} \left[\frac{e^{\lambda b_i} - e^{\lambda a_i}}{b_i - a_i} - \frac{e^{\lambda b_i} - e^{\lambda c}}{b_i - c} \right]$$

where $c = \langle z, v_0 \rangle$ □

It follows that, $\mathbb{E}[X] = \frac{d}{d\lambda} M_P(\lambda z)$ evaluated at $\lambda = 0$ and $\mathbb{E}[X^2] = \frac{d^2}{d\lambda^2} M_P(\lambda z)$ evaluated at $\lambda = 0$. Then,

$$\int_P \langle z,p\rangle \, dp = \frac{d}{d\lambda} M_P(\lambda z)\bigg|_{\lambda=0}, \quad \int_P \langle z,p\rangle^2 \, dp = \frac{d^2}{d\lambda^2} M_P(\lambda z)\bigg|_{\lambda=0}$$

and $\mathrm{Vol}(P) = \int_P dp = \lim_{\lambda\to 0} M_P(\lambda z)$.

Now, that we have a form of $M_P(\lambda z)$ in Theorem 5.3.1 we seek to convert the first form of $M_P(\lambda z)$ into the form $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_P \langle z,p\rangle^k \, dp$ so we can compute the moments $\int_P \langle z,p\rangle^n \, dp$.

Theorem 5.3.2. *The* ***second form*** *of the Moment Generating Function (MGF) of a polygon P with m sides under the triangulation mentioned in Chapter 4 is given by*

$$M_P(\lambda z) = \sum_{s=0}^{\infty} \frac{\lambda^s}{s!} \frac{1}{(s+1)(s+2)} \sum_{i=1}^{m-2} \frac{|\det A_i|}{(a_i - c)} \left[\frac{b_i^{s+2} - a_i^{s+2}}{b_i - a_i} - \frac{b_i^{s+2} - c^{s+2}}{b_i - c} \right]$$

where $a_i = \langle z, v_i\rangle, b_i = \langle z, v_{i+1}\rangle, c = \langle z, v_0\rangle, \lambda \in \mathbb{R}, z \in \mathbb{R}^2$

Proof. As derived in Theorem 5.3.1,

$$M_P(\lambda z) = \sum_{i=1}^{m-2} \frac{|\det A_i|}{\lambda^2(a_i - c)} \left[\frac{e^{\lambda b_i} - e^{\lambda a_i}}{b_i - a_i} - \frac{e^{\lambda b_i} - e^{\lambda c}}{b_i - c} \right]$$

Next, we taylor expand all the exponential terms with λ,

$$M_P(\lambda z) = \sum_{i=1}^{m-2} \frac{|\det A_i|}{\lambda^2(a_i - c)} \left[\sum_{k=0}^{\infty} \frac{\lambda^k (b_i^k - a_i^k)}{k!(b_i - a_i)} - \sum_{k=0}^{\infty} \frac{\lambda^k (b_i^k - c^k)}{k!(b_i - c)} \right]$$

$$= \sum_{i=1}^{m-2} \frac{|\det A_i|}{\lambda^2(a_i - c)} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left[\frac{(b_i^k - a_i^k)}{(b_i - a_i)} - \frac{(b_i^k - c^k)}{(b_i - c)} \right]$$

Notice, for $k = 0, 1 : \frac{(b_i^k - a_i^k)}{(b_i - a_i)} - \frac{(b_i^k - c^k)}{(b_i - c)} = 0$ hence the summation $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left[\frac{(b_i^k - a_i^k)}{(b_i - a_i)} - \frac{(b_i^k - c^k)}{(b_i - c)} \right]$ effectively starts at $k = 2$. Hence,

$$M_P(\lambda z) = \sum_{i=1}^{m-2} \frac{|\det A_i|}{\lambda^2(a_i - c)} \sum_{k=2}^{\infty} \frac{\lambda^k}{k!} \left[\frac{(b_i^k - a_i^k)}{(b_i - a_i)} - \frac{(b_i^k - c^k)}{(b_i - c)} \right]$$

If we let $s + 2 = k$ and swap the order of summation,

$$= \sum_{s=0}^{\infty} \frac{\lambda^s}{s!} \left[\frac{1}{(s+1)(s+2)} \sum_{i=1}^{m-2} \frac{|\det A_i|}{(a_i - c)} \left[\frac{(b_i^{s+2} - a_i^{s+2})}{(b_i - a_i)} - \frac{(b_i^{s+2} - c^{s+2})}{(b_i - c)} \right] \right]$$

□

Remark 5.3.3. Notice, that one quick application of the MGF is computing the volume of P. Recall, in Section 5.1, we showed that $Vol(P) = \frac{1}{2}\sum_{i=1}^{m-2} |\det A_i|$. Well now we will attempt to rederive the same equation using the MGF.

$$\text{Vol}(P) = \int_P dp = \lim_{\lambda\to 0} M_P(\lambda z)$$

As you take $\lambda \to 0$ only $s = 0$ term survives

$$= \frac{1}{(1)(2)} \sum_{i=1}^{m-2} \frac{|\det A_i|}{(a_i - c)} \left[\frac{b_i^2 - a_i^2}{b_i - a_i} - \frac{b_i^2 - c^2}{b_i - c}\right]$$

$$= \frac{1}{2} \sum_{i=1}^{m-2} \frac{|\det A_i|}{(a_i - c)} [b_i + a_i - (b_i + c)] = \frac{1}{2} \sum_{i=1}^{m-2} |\det A_i|$$

as desired.

Corollary 5.3.4. *Given a polygon P with m sides under the triangulation mentioned in Chapter 4, the n-moments are given by*

$$\int_P \langle z, p\rangle^k \, dp = \frac{1}{(s+1)(s+2)} \sum_{i=1}^{m-2} \frac{|\det A_i|}{(a_i - c)} \left[\frac{(b_i^{s+2} - a_i^{s+2})}{(b_i - a_i)} - \frac{(b_i^{s+2} - c^{s+2})}{(b_i - c)}\right]$$

where $a_i = \langle z, v_i\rangle, b_i = \langle z, v_{i+1}\rangle, c = \langle z, v_0\rangle, z \in \mathbb{R}^2$

Proof. By definition we define the moment generating function as

$$M_P(\lambda z) = \int_P e^{\lambda\langle z,p\rangle} \, dp$$

Expanding the exponential term in its taylor series and by the absolute convergence of this series we swap the order of summation,

$$= \int_P \sum_{k=0}^{\infty} \frac{\lambda^k \langle z, p\rangle^k}{k!} \, dp = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_P \langle z, p\rangle^k \, dp$$

According to the second form of $M_P(\lambda z)$,

$$\int_P \langle z, p\rangle^k \, dp = \frac{1}{(s+1)(s+2)} \sum_{i=1}^{m-2} \frac{|\det A_i|}{(a_i - c)} \left[\frac{(b_i^{s+2} - a_i^{s+2})}{(b_i - a_i)} - \frac{(b_i^{s+2} - c^{s+2})}{(b_i - c)}\right]$$

$\square$

Chapter 6

Computing the First and Second Derivatives of the Isotropic Ratio in terms of the vertices in $\mathbb{R}^2$

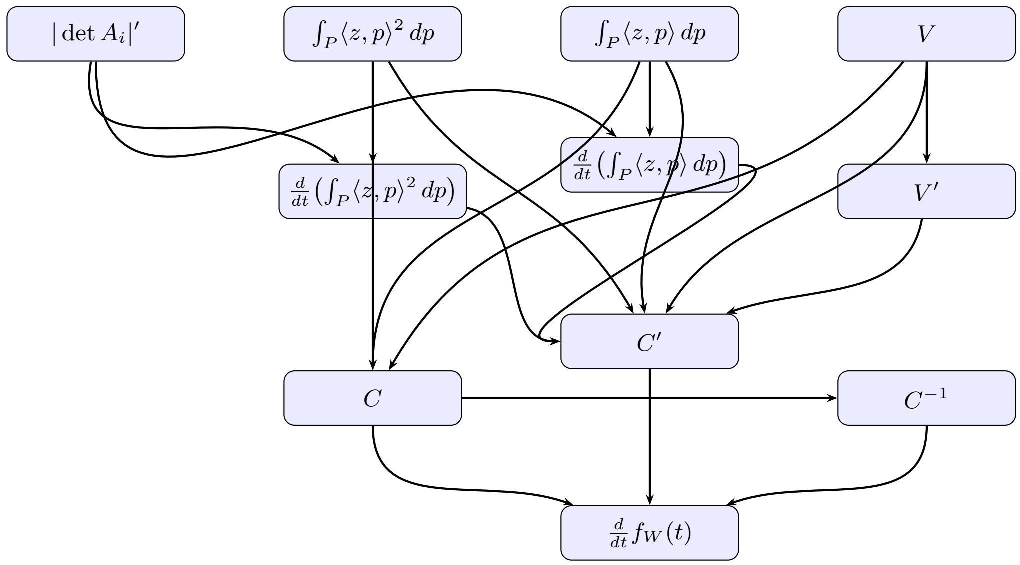
In this chapter we utilize the moment generating function as a systematic method to compute all moments of P, which are precisely the quantities needed to express $\frac{d}{dt}f_W(t)$, $\frac{d^2}{dt^2}f_W(t)$.

6.1 Computing $\frac{d}{dt}f_W(t)$: Assembling the First Derivative

Recall from Lemma 3.0.1,

$$\frac{d}{dt}f_W(t) = \frac{\mathrm{Vol}(P_{t,W})^{2n+1}}{\det \bar{C}(P_{t,W})}\left[(2n+2)\frac{d}{dt}\mathrm{Vol}(P_{t,W}) - \mathrm{Vol}(P_{t,W})\mathrm{Tr}\left[\bar{C}(P_{t,W})^{-1}\frac{d}{dt}\bar{C}(P_{t,W})\right]\right]$$

The dependency chain for $\frac{d}{dt}f_W(t)$ is then summarized in the diagram below:



Below we compute all the building blocks of $\frac{d}{dt} f_W(t)$.

6.1.1 Computing $\frac{d}{dt} v_i$ and $\frac{d^2}{dt^2} v_i$

Lemma 6.1.1. *Let $A, B \in \mathbb{R}^{n\times n}, A \in GL_n(\mathbb{R})$ then $\frac{d}{dt}(A + tB)^{-1}|_{t=0} = -A^{-1}BA^{-1}$*

Proof. Let $Z(t) = (A + tB)^{-1}$, so that $(A + tB)Z(t) = I$. Differentiating both sides with respect to t:

$$B\, Z(t) + (A + tB)\, Z'(t) = 0.$$

Evaluating at $t = 0$ gives $B\, Z(0) + A\, Z'(0) = 0$, i.e. $BA^{-1} + AZ'(0) = 0$. Multiplying on the left by A^{-1}:

$$Z'(0) = -A^{-1}BA^{-1}.$$

$\square$

Lemma 6.1.2. *Let $N_i = \begin{bmatrix} -n_i- \\ \\ -n_{i+1}- \end{bmatrix}, W_i = \begin{bmatrix} -w_i- \\ \\ -w_{i+1}- \end{bmatrix}$ where n_i is the outer normal vector to facet F_i and w_i is the perturbation assigned to F_i. Then the* ***first and second derivative of the velocity of vertex i*** *is given by*

$$\left.\frac{dv_i}{dt}\right|_{t=0} = -N_i^{-1} W_i \, v_i \tag{6.1}$$

$$\left.\frac{d^2 v_i}{dt^2}\right|_{t=0} = 2\, N_i^{-1} W_i N_i^{-1} W_i \, v_i \tag{6.2}$$

Proof. 1. Let $N_i(t) = N_i + tW_i$ as the normal outer vectors of each facet are being perturbed with respect to t by the set of perturbations W. Notice, each vertex v_i is the intersection of two consecutive facets, so it satisfies the system of equations for all $t, 0 \leq t \leq 1$

$$N_i(t)v_i(t) = c, c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Since N_i is invertible and by product rule,

$$v_i(t) = N_i(t)^{-1}c \Rightarrow \frac{d}{dt}v_i(t) = \frac{d}{dt}N_i(t)^{-1}c + N_i(t)^{-1}\frac{d}{dt}c$$

$\frac{d}{dt}c = 0$ since c is a constant vector so

$$\frac{d}{dt}v_i(t) = \frac{d}{dt}N_i(t)^{-1}c$$

Now, by Lemma 6.1.1 where $N_i(t)^{-1} = (N_i + tW_i)^{-1} \Rightarrow \frac{d}{dt}N_i(t)^{-1}\Big|_{t=0} = -N_i^{-1}W_iN_i^{-1}$, so

$$\frac{d}{dt}v_i(t)\Big|_{t=0} = -N_i^{-1}W_iN_i^{-1}c$$

But $N_i^{-1}c = v_i$,

$$= -N_i^{-1}W_iv_i$$

2. Given $\frac{d}{dt}v_i(t)\Big|_{t=0}$ we attain $W_iv_i + N_i\frac{dv_i}{dt} = 0$. Upon differentiating this expression we find

$$\frac{d}{dt}\left[W_iv_i + N_i\frac{dv_i}{dt}\right]\Bigg|_{t=0} = 0 \Rightarrow (0 + W_i\frac{dv_i}{dt}\Big|_{t=0}) + (W_i\frac{dv_i}{dt}\Big|_{t=0} + N_i\frac{d^2v_i}{dt^2}\Big|_{t=0}) = 0$$

$$\Rightarrow -2N_i^{-1}W_i\frac{dv_i}{dt}\Big|_{t=0} = \frac{d^2v_i}{dt^2}\Big|_{t=0} \Rightarrow 2N_i^{-1}W_iN_i^{-1}W_iv_i = \frac{d^2v_i}{dt^2}\Big|_{t=0}$$

as desired.

□

Remark 6.1.3. 1. W_i need not be invertible; only $N_i(t)$ (and in particular $N_i = N_i(0)$) is required to be invertible, which holds as long as the two facets at vertex v_i are not parallel.

2. c need not be $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the value of c depends on how the support functions for each facet is

defined in the beginning. However, as we will see later since c is always a constant vector, under differentiation c makes zero contribution to the derivatives.

6.1.2 Computing V, V', and $|\det A_i|'$

As shown in both Section 5.1 and Remark 5.3.3:

$$V = \frac{1}{2}\sum_{i=1}^{m-2} |\det A_i| \tag{6.3}$$

It then follows by linearity of differentiation that

$$V' = \frac{1}{2}\sum_{i=1}^{m-2} |\det A_i|' \tag{6.4}$$

where

$$|\det A_i|' = (\det A_i)\mathrm{tr}(A_i^{-1}A_i') \quad \text{where } A_i' = \begin{bmatrix} | & | \\ \frac{d(v_{i+1}-v_0)}{dt} & \frac{d(v_i - v_0)}{dt} \\ | & | \end{bmatrix} \tag{6.5}$$

Proof. Applying the chain rule gives $|\det A_i|' = \mathrm{sgn}(\det A_i) \cdot (\det A_i)'$.

Given the clockwise orientation of P, $\det \begin{bmatrix} | & | \\ v_i - v_0 & v_{i+1} - v_0 \\ | & | \end{bmatrix} < 0 \Rightarrow \det A_i > 0 \Rightarrow \mathrm{sgn}(\det A_i) = 1$

By Jacobi's formula, $(\det A_i)' = \det A_i \cdot \mathrm{tr}(A_i^{-1}A_i')$, hence

$$|\det A_i|' = (\det A_i)\mathrm{tr}(A_i^{-1}A_i')$$

□

6.1.3 Computing $\int_P \langle z, p\rangle\, dp$, $\int_P \langle z, p\rangle^2\, dp$

Using our n-moment formula in Corollary 5.3.4, in particular for $s = 1$ and $s = 2$: we can compute $\int_P \langle z, p\rangle\, dp$, $\int_P \langle z, p\rangle^2\, dp$ where $a_i = \langle z, v_i\rangle, b_i = \langle z, v_{i+1}\rangle, c = \langle z, v_0\rangle, z \in \mathbb{R}^2$

$$\int_P \langle z, p\rangle\, dp = \frac{1}{6}\sum_{i=1}^{m-2} |\det A_i|(a_i + b_i + c) \tag{6.6}$$

Proof. By Corollary 5.3.4,

$$\int_P \langle z, p\rangle \, dp = \frac{1}{(1+1)(1+2)} \sum_{i=1}^{m-2} \frac{|\det A_i|}{a_i - c} \left[\frac{b_i^3 - a_i^3}{b_i - a_i} - \frac{b_i^3 - c^3}{b_i - c} \right]$$

$$= \frac{1}{6} \sum_{i=1}^{m-2} \frac{|\det A_i|}{a_i - c} \left[(b_i^2 + a_i b_i + a_i^2) - (b_i^2 + b_i c + c^2) \right]$$

$$= \frac{1}{6} \sum_{i=1}^{m-2} \frac{|\det A_i|}{a_i - c} \left[(a_i + b_i + c)(a_i - c) \right] = \frac{1}{6} \sum_{i=1}^{m-2} |\det A_i| \left[(a_i + b_i + c) \right]$$

□

$$\int_P \langle z, p\rangle^2 \, dp = \frac{1}{12} \sum_{i=1}^{m-2} |\det A_i| \left[(a_i^2 + b_i^2 + c^2) + (b_i a_i + c(b_i + a_i)) \right] \tag{6.7}$$

Proof. By Corollary 5.3.4,

$$\int_P \langle z, p\rangle^2 \, dp = \frac{1}{(1+2)(2+2)} \sum_{i=1}^{m-2} \frac{|\det A_i|}{a_i - c} \left[\frac{b_i^4 - a_i^4}{b_i - a_i} - \frac{b_i^4 - c^4}{b_i - c} \right]$$

$$= \frac{1}{12} \sum_{i=1}^{m-2} \frac{|\det A_i|}{(a_i - c)} \left[b_i^3 + a_i b_i^2 + b_i a_i^2 + a_i^3 - [b_i^3 + c b_i^2 + b_i c^2 + c^3] \right]$$

$$= \frac{1}{12} \sum_{i=1}^{m-2} \frac{|\det A_i|}{(a_i - c)} \left[b_i^2 (a_i - c) + b_i (a_i^2 - c^2) + (a_i^2 - c^3) \right]$$

$$= \frac{1}{12} \sum_{i=1}^{m-2} |\det A_i| \left[(a_i^2 + b_i^2 + c^2) + (b_i a_i + c(b_i + a_i)) \right]$$

□

6.1.4 Computing $(\int_P \langle z, p\rangle \, dp)', (\int_P \langle z, p\rangle^2 \, dp)'$

Holding the same definitions of a_i, b_i, c

$$\frac{d}{dt} \int_P \langle z, p\rangle \, dp = \frac{1}{6} \sum_{i=1}^{m-2} (\det A_i)(\operatorname{tr}(A_i^{-1} A_i') S_i + S_i') \tag{6.8}$$

where $a_i' = \langle z, \frac{d}{dt} v_i\rangle, b_i' = \langle z, \frac{d}{dt} v_{i+1}\rangle, c' = \langle z, \frac{d}{dt} v_0\rangle, S_i = (a_i + b_i + c), S_i' = (a_i' + b_i' + c')$

Proof. Recall, $\int_P \langle z, p\rangle \, dp = \frac{1}{6} \sum_{i=1}^{m-2} |\det A_i| (a_i + b_i + c)$

By Chain Rule,

$$\frac{d}{dt}\int_P \langle z,p\rangle\,dp = \frac{1}{6}\sum_{i=1}^{m-2}\left[|\det A_i|'(a_i+b_i+c)+|\det A_i|(a_i'+b_i'+c')\right]$$

By Jacobi's Formula and since $\operatorname{sgn}(\det A_i)=1$

$$= \frac{1}{6}\sum_{i=1}^{m-2}(\det A_i)(\operatorname{tr}(A_i^{-1}A_i')(a_i+b_i+c)+(a_i'+b_i'+c')$$

□

$$\frac{d}{dt}\int_P \langle z,p\rangle^2\,dp = \frac{1}{12}\sum_{i=1}^{m-2}(\det A_i)[\operatorname{tr}(A_i^{-1}A_i')Q_i+Q_i'] \tag{6.9}$$

where $Q_i=(a_i^2+b_i^2+c^2)+(b_ia_i+cb_i+a_ic)$ and $Q_i'=2(a_ia_i'+b_ib_i'+cc')+b_i'(a_i+c)+a_i'(b_i+c)+c'(b_i+a_i)$

Proof. Recall, $\int_P\langle z,p\rangle^2\,dp = \frac{1}{12}\sum_{i=1}^{m-2}|\det A_i|\left[(a_i^2+b_i^2+c^2)+(b_ia_i+c(b_i+a_i))\right]$
Let $Q_i=(a_i^2+b_i^2+c^2)+(b_ia_i+cb_i+a_ic)$. By Chain Rule,

$$\frac{d}{dt}\int_P \langle z,p\rangle^2\,dp = \frac{1}{12}\sum_{i=1}^{m-2}\left[|\det A_i|'Q_i+|\det A_i|Q_i'\right]$$

where,

$$Q_i'=2(a_ia_i'+b_ib_i'+cc')+b_i'(a_i+c)+a_i'(b_i+c)+c'(b_i+a_i)$$

□

6.1.5 Computing $\overline{C}^{-1}(P_{t,W})$

Recall that

$$\bar{C}(P_{t,w}) = \operatorname{Vol}(P_{t,w})\int_P\langle z,p\rangle^2\,dp - \left(\int_P\langle z,p\rangle\,dp\right)^2 \tag{6.10}$$

We can compute $\overline{C}(P_{t,W})$ given that we have $Vol(P_{t,W})$, $\int_P\langle z,p\rangle\,dp$, $\int_P\langle z,p\rangle^2\,dp$ from Eq. (6.3)(6.6)(6.7). It follows that $\overline{C}^{-1}(P_{t,W})$ is solvable as well by simply taking the inverse of a 2×2 matrix.

6.1.6 Computing $\frac{d}{dt}\bar{C}(P_{t,w})$

From Eq. (6.10), differentiating with respect to t:

$$\frac{d}{dt}\bar{C}(P_{t,w}) = \underbrace{\underbrace{V'}_{\text{Eq.4}}\int_{P_t}\langle z,p\rangle^2\,dp}_{\text{Eq.7}} + \underbrace{V}_{\text{Eq.3}}\underbrace{\left(\int_{P_t}\langle z,p\rangle^2\,dp\right)'}_{\text{Eq.9}} - 2\underbrace{\int_{P_t}\langle z,p\rangle\,dp}_{\text{Eq.6}}\underbrace{\left(\int_{P_t}\langle z,p\rangle\,dp\right)'}_{\text{Eq.8}} \tag{6.11}$$

By Eq. (6.3), (6.4), (6.6), (6.7), (6.8), (6.9), we can now compute $\frac{d}{dt}\overline{C}(P_{t,W})$.

6.1.7 Computing $\frac{d}{dt}f_W(t)$

Substituting the computed components from Section 6.1,

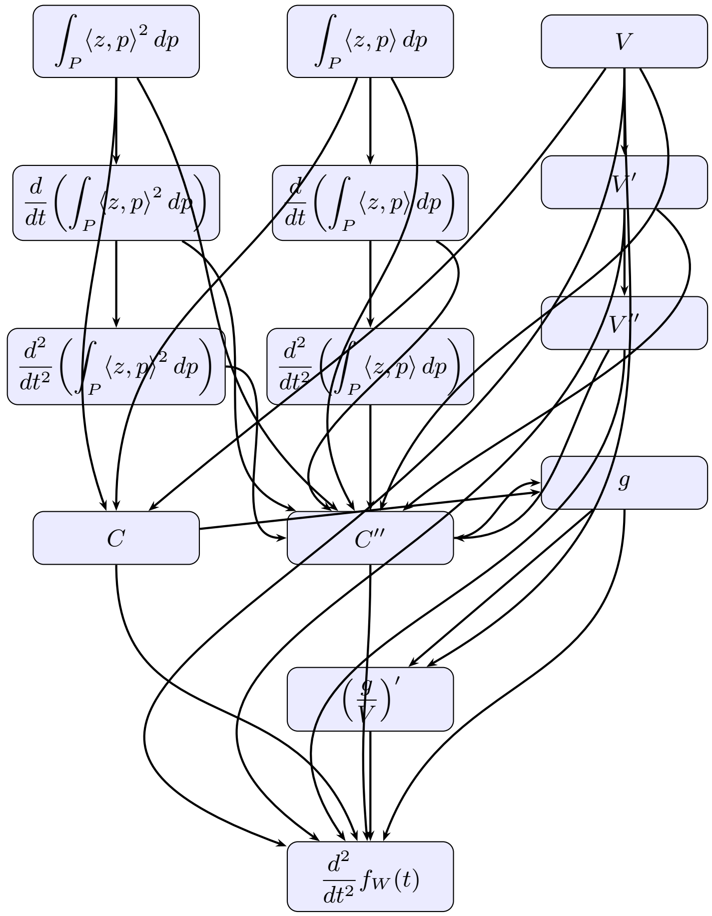
$$\frac{d}{dt}f_W(t) = \frac{\overbrace{V}^{\text{Eq.3}}{}^{2n+1}}{\underbrace{\det\bar{C}}_{\text{Eq.10}}}\left[(2n+2)\underbrace{V'}_{\text{Eq.4}} - \underbrace{V}_{\text{Eq.3}}\ \mathrm{tr}\left[\underbrace{\bar{C}(P_{t,w})^{-1}}_{\text{from Eq.10}}\cdot\underbrace{\frac{d}{dt}\bar{C}(P_{t,w})}_{\text{Eq.11}}\right]\right] \tag{6.12}$$

6.2 Computing $\frac{d^2}{dt^2}f_W(t)$: Assembling the Second Derivative

Recall from Lemma 3.0.2,

$$\frac{d^2}{dt^2}f_W(t) = \frac{V^{2n}}{D}\left[((2n+1)\frac{d}{dt}V - g)((2n+2)\frac{d}{dt}V - g) + (2n+2)V\frac{d^2}{dt^2}V - V^2\frac{d}{dt}\frac{g}{V} - g\frac{d}{dt}V)\right]$$

The dependency diagram for $\frac{d^2}{dt^2}f_W(t)$ is then given below:



Notice, to assemble $\frac{d^2}{dt^2} f_W(t)$ we now need the following new pieces:

$$V'',\ \left(\frac{g}{V}\right)',\ g = V\,\mathrm{tr}[\bar{C}^{-1}\bar{C}'],\ (\frac{g}{V})',\ \left(\int_P \langle z,p\rangle^2\,dp\right)'',\ \left(\int_P \langle z,p\rangle\,dp\right)'',\ C''$$

6.2.1 Computing $|\det A_i|''$ and V''

This is one of the components that appears up in many of our later terms, so we compute it first.

$$|\det A_i|'' = \det A_i\left[(\mathrm{tr}(A_i^{-1}A_i'))^2 + \mathrm{tr}\left(-A_i^{-1}A_i'A_i^{-1}A_i' + A_i^{-1}A_i''\right)\right] \tag{6.13}$$

where

$$A_i' = \begin{bmatrix} | & | \\ \dfrac{d(v_{i+1}-v_0)}{dt} & \dfrac{d(v_i - v_0)}{dt} \\ | & | \end{bmatrix}, \qquad A_i'' = \begin{bmatrix} | & | \\ \dfrac{d^2(v_{i+1}-v_0)}{dt^2} & \dfrac{d^2(v_i - v_0)}{dt^2} \\ | & | \end{bmatrix}$$

Proof. Recall by Eq. 6.5,

$$|\det A_i|' = (\det A_i)\mathrm{tr}(A_i^{-1}A_i')$$

Under product rule,

$$|\det A_i|'' = (\det A_i)'\mathrm{tr}(A_i^{-1}A_i') + (\det A_i)\mathrm{tr}(A_i^{-1}A_i')'$$

By Jacobi's Formula,

$$= \det A_i \mathrm{tr}(A_i^{-1}A_i')^2 + \det A_i \mathrm{tr}(A_i^{-1}A_i')'$$

Since tr (trace) is linear we can push the differentiation inside and use product rule,

$$\mathrm{tr}(A_i^{-1}A_i')' = \mathrm{tr}((A_i^{-1})'A_i' + A_i^{-1}A_i'')$$

Notice, A_i is invertible as all outer normal vectors are linearly independent. Applying Lemma 6.1.1,on $A_i(t) = A_i + tA_i'$ we find $(A_i(t))^{-1}|_{t=0} = -A_i^{-1}A_i'A_i^{-1}$. Hence,

$$|\det A_i|'' = \det A_i \left[(\mathrm{tr}(A_i^{-1}A_i'))^2 + \mathrm{tr}\left(-A_i^{-1}A_i'A_i^{-1}A_i' + A_i^{-1}A_i''\right)\right]$$

□

By differentiating V' term by term we arrive at

$$V'' = \frac{1}{2}\sum_{i=1}^{m-2} |\det A_i|'' \tag{6.14}$$

where $|\det A_i|''$ is given by Eq. (6.13)

6.2.2 Computing $(\int_P \langle z,p\rangle\, dp)''$, $(\int_P \langle z,p\rangle^2\, dp)''$

Let $p_i = \mathrm{tr}(\mathrm{A_i^{-1}A_i'})$ and adopt the notations S_i, S_i', Q_i, Q_i' in Eq. 6.8, 6.9,

$$(\int_P \langle z,p\rangle\, dp)'' = \frac{1}{6}\sum_{i=1}^{m-2} (\det A_i)\left[2p_iS_i' + (p_i^2 + p_i')S_i + S_i''\right] \tag{6.15}$$

Proof. Recall by Eq. 6.8,

$$(\int_P \langle z,p\rangle\, dp)' = \frac{1}{6}\sum_{i=1}^{m-2} (\det A_i)(p_iS_i + S_i')$$

Then by chain rule and applying product rule on p_iS_i,

$$(\int_P \langle z,p\rangle\, dp)'' = \frac{1}{6}\sum_{i=1}^{m-2}[(\det A_i)'(p_iS_i + S_i') + (\det A_i)(p_i'S_i + p_iS_i' + S_i'')]$$

As proved in many situations above with Jacobi's Formula, $(\det A_i)' = \det A_i \mathrm{tr}(A_i^{-1}A_i') = \det A_i p_i$,

$$= \frac{1}{6}\sum_{i=1}^{m-2}[(\det A_i)(p_i)(p_iS_i + S_i') + (\det A_i)(p_i'S_i + p_iS_i' + S_i'')]$$

Expanding all terms and regrouping we attain

$$(\int_P \langle z,p\rangle\, dp)'' = \frac{1}{6}\sum_{i=1}^{m-2}(\det A_i)\left[2p_iS_i' + (p_i^2 + p_i')S_i + S_i''\right]$$

where $S_i'' = (a_i'' + b_i'' + c'')$ □

$$(\int_P \langle z,p\rangle^2\, dp)'' = \frac{1}{12}\sum_{i=1}^{m-2}(\det A_i)(Q_i(p_i^2 + p_i') + 2p_iQ_i' + Q_i'') \tag{6.16}$$

where $Q_i'' = (a_i''T_i + b_i''U_i + c''V_i) + (a_i'T_i' + b_i'U_i' + c'V_i')$, $T_i = (b_i + c + 2a_i)$, $U_i = (a_i + c + 2b_i)$, $V_i = (b_i + a_i + 2c)$ and p_i' was computed in the proof of Eq. 6.13

Proof. Recall by Eq. 6.9,

$$(\int_P \langle z,p\rangle^2\, dp)' = \frac{1}{12}\sum_{i=1}^{m-2}(\det A_i)[\mathrm{tr}(A_i^{-1}A_i')Q_i + Q_i']$$

where $Q_i = (a_i^2+b_i^2+c^2)+(b_ia_i+cb_i+a_ic)$ and $Q_i' = 2(a_ia_i'+b_ib_i'+cc')+b_i'(a_i+c)+a_i'(b_i+c)+c'(b_i+a_i)$
Then again by chain rule and product rule,

$$(\int_P \langle z,p\rangle^2\, dp)'' = \frac{1}{12}\sum_{i=1}^{m-2}(\det A_i)'(p_iQ_i + Q_i') + (\det A_i)(p_i'Q_i + p_iQ_i' + Q_i'')$$

$$= \frac{1}{12}\sum_{i=1}^{m-2}(\det A_i)(p_i^2Q_i + p_iQ_i' + p_i'Q_i + p_iQ_i' + Q_i'')$$

$$= \frac{1}{12}\sum_{i=1}^{m-2}(\det A_i)(Q_i(p_i^2 + p_i') + 2p_iQ_i' + Q_i'')$$

Now, we compute Q_i''.

$$Q_i' = 2(a_i a_i' + b_i b_i' + cc') + b_i'(a_i + c) + a_i'(b_i + c) + c'(b_i + a_i)$$

$$\Rightarrow Q_i'' = 2[(a_i')^2 + a_i(a_i'') + (b_i')^2 + b_i(b_i'') + (c')^2 + c(c'')] + b_i''(a_i + c) +$$

$$b_i'(a_i' + c') + a_i''(b_i + c) + a_i'(b_i' + c') + c''(b_i + a_i) + c'(b_i' + a_i')$$

$$= a_i''(b_i+c+2a_i)+b_i''(a_i+c+2b_i)+c''(b_i+a_i+2c)+a_i'(b_i'+c'+2a_i')+b_i'(a_i'+c'+2b_i')+c'(b_i'+a_i'+2c')$$

If we let $T_i = (b_i + c + 2a_i)$, $U_i = (a_i + c + 2b_i)$, $V_i = (b_i + a_i + 2c)$ then it follows that $Q_i'' = (a_i''T_i + b_i''U_i + c''V_i) + (a_i'T_i' + b_i'U_i' + c'V_i')$

□

6.2.3 Computing C''

Now that we have computed V'', $(\int_P \langle z,p\rangle^2\, dp)''$, $(\int_P \langle z,p\rangle\, dp)''$ we are ready to assemble C''

$$\bar{C}'' = V''\int_P \langle z,p\rangle^2\, dp + 2V'(\int_P \langle z,p\rangle^2\, dp)' + V(\int_P \langle z,p\rangle^2\, dp)'' - 2[((\int_P \langle z,p\rangle\, dp)')^2 + (\int_P \langle z,p\rangle\, dp)(\int_P \langle z,p\rangle\, dp)''] \tag{6.17}$$

Proof. Let $X_1 = \int_P \langle z,p\rangle^2\, dp$ and $X_2 = \int_P \langle z,p\rangle\, dp$. Then, Eq 6.11 tells us that

$$\bar{C}' = \underbrace{V'}_{\text{Eq.4}}\underbrace{X_1}_{\text{Eq.7}} + \underbrace{V}_{\text{Eq.3}}\underbrace{X_1'}_{\text{Eq.9}} - 2\underbrace{X_2}_{\text{Eq.6}}\underbrace{X_2'}_{\text{Eq.8}}$$

By applying product rule on each term we find

$$\bar{C}'' = V''X_1 + V'X_1' + V'X_1' + VX_1'' - 2(X_2'X_2' + X_2X_2'')$$

After combining like terms,

$$= V''X_1 + 2V'X_1' + VX_1'' - 2((X_2')^2 + X_2X_2'')$$

Substituting $X_1 = \int_P \langle z,p\rangle^2\, dp$ and $X_2 = \int_P \langle z,p\rangle\, dp$ back in, we get

$$\bar{C}'' = V''\int_P \langle z,p\rangle^2\, dp + 2V'(\int_P \langle z,p\rangle^2\, dp)' + V(\int_P \langle z,p\rangle^2\, dp)'' - 2[((\int_P \langle z,p\rangle\, dp)')^2 + (\int_P \langle z,p\rangle\, dp)(\int_P \langle z,p\rangle\, dp)'']$$

$\square$

6.2.4 Computing g and $(\frac{g}{V})'$

By construction we let

$$g = V\,\mathrm{tr}[\bar{C}^{-1}\bar{C}'] \tag{6.18}$$

where V, $\bar{C}^{-1}, \bar{C}'$ were all computed in Section 6.1.

$$\left(\frac{g}{V}\right)' = \mathrm{tr}\big[-\bar{C}^{-1}\bar{C}'\bar{C}^{-1}\bar{C}' + \bar{C}^{-1}\bar{C}''\big] \tag{6.19}$$

where $\bar{C}''$ was computed in Eq. 6.17

Proof.

$$(\frac{g}{V})' = \mathrm{tr}[\bar{C}^{-1}\bar{C}']' = \mathrm{tr}[(\bar{\mathrm{C}}^{-1}\bar{\mathrm{C}}')'] = \mathrm{tr}[(\bar{C}^{-1})'\bar{C}' + \bar{C}^{-1}\bar{C}'']$$

Notice, since $\bar{C}$ is symmetric positive definite it is invertible. Then by Lemma 6.1.1 where $\bar{C}(t)^{-1} = (\bar{C} + t\bar{C}')^{-1} \Rightarrow \frac{d}{dt}\bar{C}(t)^{-1}\Big|_{t=0} = -\bar{C}^{-1}\bar{C}'\bar{C}^{-1}$ and substituting it into $(\frac{g}{V})'$ we get

$$(\frac{g}{V})' = \mathrm{tr}[-\bar{C}^{-1}\bar{C}'\bar{C}^{-1}\bar{C} + \bar{C}^{-1}\bar{C}'']$$

$\square$

6.2.5 Computing $\frac{d^2}{dt^2}f_W(t)$

Substituting the computed components from Section 6.2,

$$\frac{d^2}{dt^2}f_W(t) = \frac{V^{2n}}{D}\left[((2n+1)\frac{d}{dt}V - g)((2n+2)\frac{d}{dt}V - g) + (2n+2)V\underbrace{\frac{d^2}{dt^2}V}_{\text{Eq. 6.14}} - V^2\underbrace{\frac{d}{dt}\frac{g}{V}}_{\text{Eq. 6.19}} - \underbrace{g}_{\text{Eq. 6.18}}\frac{d}{dt}V)\right] \tag{6.20}$$

where $D = \det\bar{C}$.

Chapter 7

Example Computations of Isotropic Positions, $\frac{d}{dt}f_W(t)$ and $\frac{d^2}{dt^2}f_W(t)$

In this chapter we will compute $\frac{d}{dt}f_W(t)$, $\frac{d^2}{dt^2}f_W(t)$ by hand on a square to help us verify whether these equations have been implemented correctly in code. Going through such computations for a single example helps us also see why we would want to build software to help us compute such values. Before diving straight into $\frac{d}{dt}f_W(t)$, $\frac{d^2}{dt^2}f_W(t)$, we will compute the isotropic position of any given square and triangle.

7.1 Computing Isotropic Positions

7.1.1 Example: Isotropic Position of a Square

Here we compute the isotropic position of any given square under the affine map $T(P) = C(P)^{-1/2}(x-b_P)$.

Setup. Let $V(P) = \{(a,a), (a,-a), (-a,-a), (-a,a)\}$, so $\mathrm{Vol}(P) = 4a^2$ and $b_P = (0,0)$ by symmetry (since $\int_{-a}^{a} x\,dx = \int_{-a}^{a} y\,dy = 0$).

Computing $C(P)$. Recall

$$C(P) = \frac{\int_P xx^T\,dx}{\mathrm{Vol}(P)} - \frac{\int_P x\,dx}{\mathrm{Vol}(P)} \cdot \frac{\int_P x^T\,dx}{\mathrm{Vol}(P)}.$$

Since $b_P = 0$, the second term vanishes. For the off-diagonal entries:

$$\iint_{-a}^{a} xy\,dx\,dy = \int_{-a}^{a} x\,dx \cdot \int_{-a}^{a} y\,dy = 0.$$

For the diagonal entries:

$$\iint_{-a}^{a} x^2\,dx\,dy = \int_{-a}^{a} x^2\,dx \cdot \int_{-a}^{a} dy = \frac{2a^3}{3} \cdot 2a = \frac{4a^4}{3}.$$

By symmetry the y^2 entry is the same, so

$$C(P) = \frac{1}{4a^2} \begin{bmatrix} \frac{4a^4}{3} & 0 \\ 0 & \frac{4a^4}{3} \end{bmatrix} = \frac{a^2}{3} I_2$$

where I_n is the $n \times n$ identity matrix. Hence $C(P)^{-1/2} = \frac{\sqrt{3}}{a} I_2$, and

$$T(P) = \frac{\sqrt{3}}{a} P \implies \boxed{V(T(P)) = \{(\sqrt{3}, \sqrt{3}),\ (\sqrt{3}, -\sqrt{3}),\ (-\sqrt{3}, -\sqrt{3}),\ (-\sqrt{3}, \sqrt{3})\}}$$

7.1.2 Example: Isotropic Position of a Triangle

Let $V(P) = \{(0, a), (a, 0), (0, 0)\}$, so $\mathrm{Vol}(P) = \frac{a^2}{2}$ and

$$b_P = \left(\frac{a}{2}, \frac{a}{2}\right) \text{ – (computed via the standard centroid formula) – see below.}$$

We compute b_P and $C(P)$ by direct integration over the triangle $\{(x, y) : x \geq 0,\ y \geq 0,\ x + y \leq a\}$.

Off-diagonal entry.

$$\iint xy\,dy\,dx = \int_0^a \int_0^{a-x} xy\,dy\,dx = \int_0^a x \cdot \frac{(a-x)^2}{2}\,dx = \frac{1}{2}\int_0^a x(a^2 - 2ax + x^2)\,dx = \frac{a^4}{24}.$$

First moments.

$$\iint x\,dy\,dx = \int_0^a x(a-x)\,dx = \frac{a^3}{6}, \qquad \iint y\,dy\,dx = \frac{a^3}{6} \implies b_P = \left(\frac{a}{3}, \frac{a}{3}\right).$$

Second moments (diagonal, $x = y$ case).

$$\int_0^a \int_0^{a-x} x^2\,dy\,dx = \int_0^a x^2(a-x)\,dx = \frac{a^4}{12}.$$

Hence

$$C(P)_{i=j} = \frac{a^4/12}{a^2/2} - \frac{a^2}{9} = \frac{a^2}{6} - \frac{a^2}{9} = \frac{a^2}{18}, \qquad C(P)_{i\neq j} = \frac{a^4/24}{a^2/2} - \frac{a^2}{9} = \frac{a^2}{12} - \frac{a^2}{9} = -\frac{a^2}{36}.$$

So

$$C(P) = \frac{a^2}{36}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad C(P)^{-1} = \frac{12}{a^2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \det C(P) = \frac{a^4}{36^2}\cdot 3 = \frac{a^4}{432}.$$

To find $C(P)^{-1/2}$, we diagonalize: eigenvalues of $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ are $\lambda_1 = 3, \lambda_2 = 1$, giving

$$C(P)^{-1/2} = \frac{\sqrt{12}}{a}\begin{bmatrix} \frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}-1}{2} \\ \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \end{bmatrix} \implies \boxed{V(T(P)) = \{(6-\sqrt{12}, 6+\sqrt{12}),\ (6+\sqrt{12}, 6-\sqrt{12}),\ (0,0)\}}$$

7.2 Handwritten Notes of Computing $\frac{d}{dt}f_W(t)$ and $\frac{d^2}{dt^2}f_W(t)$ on a Square

$$v_0 = (-a, a), \quad v_1 = (a, a), \quad v_2 = (a, -a), \quad v_3 = (-a, -a).$$

$$A_i = \begin{bmatrix} v_{i+1} - v_0 & v_i - v_0 \end{bmatrix}, \qquad W = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad N_i = \begin{bmatrix} -n_{i-1}- \\ -n_i- \end{bmatrix}, \qquad W_i = \begin{bmatrix} -w_{i-1}- \\ -w_i- \end{bmatrix}.$$

Conditions on the normals n_i:

$$\langle (x, a), n_0\rangle = 1\ \forall x \Rightarrow n_0 = \left(0, \tfrac{1}{a}\right),$$

$$\langle (x, -a), n_2\rangle = 1\ \forall x \Rightarrow n_2 = \left(0, -\tfrac{1}{a}\right),$$

$$\langle (a, y), n_1\rangle = 1\ \forall y \Rightarrow n_1 = \left(\tfrac{1}{a}, 0\right),$$

$$\langle (-a, y), n_3\rangle = 1\ \forall y \Rightarrow n_3 = \left(-\tfrac{1}{a}, 0\right).$$

$$A_i' = \begin{bmatrix} \frac{d(v_{i+1}-v_0)}{dt} & \frac{d(v_i-v_0)}{dt} \end{bmatrix}.$$

$$\Rightarrow \quad N_0 = \begin{bmatrix} -\frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0 & \frac{1}{a} \\ \frac{1}{a} & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & -\frac{1}{a} \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0 & -\frac{1}{a} \\ -\frac{1}{a} & 0 \end{bmatrix}.$$

$$W_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad W_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

First derivatives at $t = 0$:

$$\frac{d}{dt}v_0(t)\Big|_{t=0} = -N_0^{-1}W_0v_0 = -\begin{bmatrix} -\frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -a \\ a \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}\begin{bmatrix} 0 \\ -a \end{bmatrix} = \begin{bmatrix} 0 \\ a^2 \end{bmatrix},$$

$$\frac{d}{dt}v_1(t)\Big|_{t=0} = -N_1^{-1}W_1v_1 = \begin{bmatrix} 0 & \frac{1}{a} \\ \frac{1}{a} & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} 0 & -a \\ -a & 0 \end{bmatrix}\begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -a^2 \end{bmatrix},$$

$$\frac{d}{dt}v_2(t)\Big|_{t=0} = \frac{d}{dt}v_3(t)\Big|_{t=0} = 0.$$

$$A_i'' = \begin{bmatrix} \frac{d^2(v_{i+1}-v_0)}{dt^2} & \frac{d^2(v_i-v_0)}{dt^2} \end{bmatrix}.$$

Second derivatives at $t = 0$:

$$\frac{d^2}{dt^2}v_0(t)\Big|_{t=0} = 2(N_0^{-1}W_0)^2v_0 = 2\left(\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)^2\begin{bmatrix} -a \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(Intermediate: the product squared is $\left[\begin{smallmatrix} 0 & 0 \\ -a & 0 \end{smallmatrix}\right]^2$.)

$$\frac{d^2}{dt^2}v_1(t)\Big|_{t=0} = 2(N_1^{-1}W_1)^2v_1 = 2\left(\begin{bmatrix} -a \\ a \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)^2\begin{bmatrix} a \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

(Intermediate: product squared is $\left[\begin{smallmatrix} 0 & 0 \\ -a & 0 \end{smallmatrix}\right]^2$.)

$$\frac{d^2}{dt^2}v_2\Big|_{t=0} = \frac{d^2}{dt^2}v_3\Big|_{t=0} = 0.$$

Computing A_1, A_2 and their determinants/inverses:

$$A_1 = \begin{bmatrix} a-(-a) & a-(-a) \\ -a-a & a-a \end{bmatrix} = \begin{bmatrix} 2a & 2a \\ -2a & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -a-(-a) & a-(-a) \\ -a-a & -a-a \end{bmatrix} = \begin{bmatrix} 0 & 2a \\ -2a & -2a \end{bmatrix}.$$

$$\det A_1 = \det A_2 = 4a^2.$$

$$A_1' = \begin{bmatrix} 0-0 & 0-0 \\ 0-a^2 & -a^2-a^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -a^2 & -2a^2 \end{bmatrix}.$$

$$A_1^{-1} = \frac{1}{4a^2}\begin{bmatrix} 0 & -2a \\ 2a & 2a \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2a} \\ \frac{1}{2a} & \frac{1}{2a} \end{bmatrix}, \qquad A_2^{-1} = \frac{1}{4a^2}\begin{bmatrix} -2a & -2a \\ 2a & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2a} & -\frac{1}{2a} \\ \frac{1}{2a} & 0 \end{bmatrix}.$$

$$A_2' = \begin{bmatrix} 0-0 & 0-0 \\ 0-a^2 & 0-a^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -a^2 & -a^2 \end{bmatrix}.$$

(3) $V = \frac{1}{2}(4a^2 + 4a^2) = 4a^2.$

(4) $V' = \frac{1}{2}(-2a^3 + 2a^3) = 0. \quad p_1 = -\frac{a}{2}, \quad p_2 = \frac{a}{2}.$

$$|\det A_1|' = 4a^2 \ \mathrm{tr}\left(\begin{bmatrix} 0 & -\frac{1}{2a} \\ \frac{1}{2a} & \frac{1}{2a} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -a^2 & -2a^2 \end{bmatrix} \right) = 4a^2\left(\frac{a}{2} - a\right) = -2a^3.$$

$$|\det A_2|' = 4a^2 \ \mathrm{tr}\left(\begin{bmatrix} -\frac{1}{2a} & -\frac{1}{2a} \\ \frac{1}{2a} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -a^2 & -a^2 \end{bmatrix} \right) = 4a^2\left(\frac{a}{2} + 0\right) = 2a^3.$$

(5) $|\det A_1|' = -2a^3, \quad |\det A_2|' = 2a^3.$

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(6)

$$\int_P \langle z, p\rangle \, dp = \frac{1}{6}\sum_{i=1}^{m-2} |\det A_i| \, (a_i + b_i + c).$$

For the square ($m-2=2$ triangles):

$$(a_1+b_1+c) = \langle z, \begin{bmatrix} a \\ a \end{bmatrix} + \begin{bmatrix} a \\ -a \end{bmatrix} + \begin{bmatrix} -a \\ a \end{bmatrix} \rangle = \langle z, \begin{bmatrix} a \\ a \end{bmatrix} \rangle$$

$$(a_2+b_2+c) = \langle z, \begin{bmatrix} a \\ -a \end{bmatrix} + \begin{bmatrix} -a \\ -a \end{bmatrix} + \begin{bmatrix} a \\ a \end{bmatrix} \rangle = \langle z, \begin{bmatrix} -a \\ -a \end{bmatrix} \rangle$$

$$\Rightarrow \quad (6): \quad \frac{4a^2}{6} \langle z, \begin{bmatrix} a \\ a \end{bmatrix} + \begin{bmatrix} -a \\ -a \end{bmatrix} \rangle \ = 0.$$

(7)

$$\int_P \langle z, p \rangle^2 \, dp = \frac{1}{12} \sum_{i=1}^{m-2} |\det A_i| \Big[(a_i^2 + b_i^2 + c^2) + (b_i a_i + c(b_i + a_i)) \Big]$$

and $Q_i = \Big[(a_i^2 + b_i^2 + c^2) + (b_i a_i + c(b_i + a_i)) \Big]$

Triangle 1:

$$a_1^2 = (az_1 + az_2)^2 = a^2(z_1+z_2)^2, \quad b_1^2 = (az_1 - az_2)^2 = a^2(z_1 - z_2)^2, \quad c^2 = (-az_1 + az_2)^2 = a^2(z_1 - z_2)^2.$$

$$\Rightarrow \quad a_1^2 + b_1^2 + c^2 = a^2\big((z_1+z_2)^2 + 2(z_1-z_2)^2\big).$$

$$b_1 a_1 = (az_1 - az_2)(az_1 + az_2) = a^2(z_1^2 - z_2^2).$$

$$c(b_1 + a_1) = (-az_1 + az_2)\big[(az_1 - az_2) + (az_1 + az_2)\big] = 2a^2(-z_1 + z_2)z_1.$$

$$\Rightarrow \quad Q_1 = a^2\big((z_1+z_2)^2 + 2(z_1-z_2)^2\big) + a^2(z_1^2 - z_2^2) + 2a^2 z_1(-z_1 + z_2).$$

Expanding:

$$\begin{aligned} Q_1 &= a^2\big((z_1+z_2)^2 + 2(z_1-z_2)^2 + (z_1^2 - z_2^2) + 2z_1(-z_1+z_2)\big) \\ &= a^2\big(z_1^2 + 2z_1z_2 + z_2^2 + 2(z_1^2 - 2z_1z_2 + z_2^2) + z_1^2 - z_2^2 + 2(-z_1^2 + z_1z_2)\big) \\ &= a^2(2z_1^2 + 2z_2^2) = 2a^2(z_1^2 + z_2^2). \end{aligned}$$

Triangle 2:

$$a_2^2 = (az_1 - az_2)^2 = a^2(z_1 - z_2)^2, \quad b_2^2 = (-az_1 - az_2)^2 = a^2(z_1 + z_2)^2, \quad c^2 = a^2(z_1 - z_2)^2.$$

$$\Rightarrow \quad a_2^2 + b_2^2 + c^2 = a^2\big((z_1+z_2)^2 + 2(z_1-z_2)^2\big).$$

$$b_2 a_2 = (-az_1 - az_2)(az_1 - az_2) = -a^2(z_1^2 - z_2^2).$$

$$c(b_2 + a_2) = (-az_1 + az_2)\big[(-az_1 - az_2) + (az_1 - az_2)\big] = -2a^2(-z_1 + z_2)z_2.$$

$$\Rightarrow \quad Q_2 = a^2\big((z_1+z_2)^2 + 2(z_1-z_2)^2 - (z_1^2 - z_2^2) - 2(-z_1z_2 + z_2^2)\big) = a^2(2z_1^2 + 2z_2^2) = 2a^2(z_1^2+z_2^2).$$

$$\Rightarrow \quad Q_1 = Q_2, \qquad \text{and}$$

$$\boxed{\int_P \langle z,p\rangle^2\,dp = \frac{4a^2}{12}\big(2a^2(z_1^2+z_2^2) + 2a^2(z_1^2+z_2^2)\big) = \frac{4}{3}\,a^4\,(z_1^2+z_2^2)}$$

---

(8)

$$\left(\int_P \langle z,p\rangle\,dp\right)' = \frac{1}{6}\sum_{i=1}^{m-2}(\det A_i)\big(p_i\,S_i + S_i'\big),$$

where $p_i = \operatorname{tr}(A_i^{-1}A_i')$ (computed in steps 3,4): $p_1 = -\dfrac{a}{2}$, $p_2 = \dfrac{a}{2}$, and (computed in step 6): $S_1 = \langle z, \left[\begin{smallmatrix} a \\ a\end{smallmatrix}\right]$, $S_2 = \langle z, \left[\begin{smallmatrix} -a \\ -a\end{smallmatrix}\right]$.

$$S_1' = a_1' + b_1' + c' = \langle z, \begin{bmatrix} 0 \\ -a^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a^2 \end{bmatrix}\rangle = \langle z, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\rangle = 0.$$

$$S_2' = a_2' + b_2' + c' = \langle z, \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a^2 \end{bmatrix}\rangle = \langle z, \begin{bmatrix} 0 \\ a^2 \end{bmatrix}\rangle = a^2 z_2.$$

$$\begin{aligned} \Rightarrow \quad (8) &= \frac{4a^2}{6}\left(\Big[\Big[-\tfrac{a}{2}\Big]\,a(z_1+z_2) + 0\Big] + \Big[\Big[\tfrac{a}{2}\Big]\,(-a(z_1+z_2)) + a^2 z_2\Big]\right) \\ &= \frac{2a^2}{3}\big(-a^2(z_1+z_2) + a^2 z_2\big) = \boxed{-\frac{2}{3}\,a^4\,z_1}. \end{aligned}$$

---

(9)

$$\left(\int_P \langle z,p\rangle^2\,dp\right)' = \frac{1}{12}\sum_{i=1}^{m-2}(\det A_i)\Big[p_i\,Q_i + Q_i'\Big],$$

with Q_1, Q_2 computed in step 7.

$$(a_1a_1' + b_1b_1' + cc') = (az_1 + az_2)(-a^2z_2) + (az_1 - az_2)(0) + (-az_1 + az_2)(a^2z_2) = -2a^3z_1z_2.$$

$$b_1'(a_1 + c) = \langle z, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rangle (a_1 + c) = 0.$$

$$a_1'(b_1 + c) = (-a^2z_2)\langle z, \begin{bmatrix} a \\ -a \end{bmatrix} + \begin{bmatrix} -a \\ a \end{bmatrix} \rangle = 0.$$

$$c'(b_1 + a_1) = (a^2z_2)\left\langle z, \begin{bmatrix} a \\ -a \end{bmatrix} + \begin{bmatrix} a \\ a \end{bmatrix} \right\rangle = (a^2z_2) \cdot 2(az_1) = 2a^3z_1z_2.$$

$$\Rightarrow \quad Q_1' = 2(-2a^3z_1z_2) + 2a^3z_1z_2 = -2a^3z_1z_2.$$

$$(a_2a_2' + b_2b_2' + cc') = (az_1 - az_2)(0) + (-az_1 - az_2)(0) + (-az_1 + az_2)(a^2z_2) = -a^3(z_1z_2 - z_2^2).$$

$$b_2'(a_2 + c) = 0, \quad a_2'(b_2 + c) = 0,$$

$$c'(b_2 + a_2) = (a^2z_2)\left\langle z, \begin{bmatrix} -a \\ -a \end{bmatrix} + \begin{bmatrix} a \\ -a \end{bmatrix} \right\rangle = (a^2z_2)(-2az_2) = -2a^3z_2^2.$$

$$\Rightarrow \quad Q_2' = 2(-a^3(z_1z_2 - z_2^2)) - 2a^3z_2^2 = -2a^3z_1z_2.$$

$$\Rightarrow \quad (9) = \frac{4a^2}{12}\left[(p_1Q_1 + Q_1') + (p_2Q_2 + Q_2')\right] = \frac{a^2}{3}\left[2(-2a^3z_1z_2)\right] = \boxed{-\frac{4a^5}{3}\, z_1z_2}.$$

(using $p_1Q_1 = -p_2Q_2$, so those terms cancel)

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(10) Covariance matrix $\bar{C}(p, z)$:

$$\langle z, \bar{C}(p)z \rangle = V \int_P \langle z, p \rangle^2 \, dp - \left(\int_P \langle z, p \rangle \, dp \right)^2.$$

$$= 4a^2\left(\frac{4}{3}a^4(z_1^2+z_2^2)\right) - (0)^2 = \frac{16}{3}a^6\,\|z\|_2^2.$$

So $\bar{C} = \begin{bmatrix} \frac{16a^6}{3} & 0 \\ 0 & \frac{16a^6}{3} \end{bmatrix}$, and $\det\bar{C} = \dfrac{256}{9}a^{12}$.

(11) $\langle z, \bar{C}(p)z\rangle$:

$$= V'\int_P \langle z,p\rangle^2\,dp + V\left(\int_P \langle z,p\rangle^2\,dp\right)' - 2\int_P \langle z,p\rangle\,dp\cdot\left(\int_P \langle z,p\rangle\,dp\right)'$$

$$= (0)\left(\frac{4}{3}a^4\|z\|_2^2\right) + (4a^2)\left(-\frac{4a^5}{3}z_1z_2\right) - 2(0)\left(\int_P \langle z,p\rangle\,dp\right)' = \boxed{-\frac{16a^7}{3}\,z_1z_2}.$$

$$\bar{C}'\bar{C}^{-1} = \begin{bmatrix} 0 & -\frac{8a^7}{3} \\ -\frac{8a^7}{3} & 0 \end{bmatrix}\begin{bmatrix} \frac{3}{16a^6} & 0 \\ 0 & \frac{1}{16a^6} \end{bmatrix} \Rightarrow \operatorname{tr}(\bar{C}^{-1}C') = 0.$$

$$z^T\bar{C}'z = \alpha z_1^2 + 2\beta z_1z_2 + \gamma z_2^2.$$

where $\bar{C} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$, so $\bar{C}' = \begin{pmatrix} 0 & -\frac{8}{3}a^7 \\ -\frac{8}{3}a^7 & 0 \end{pmatrix}$, $\bar{C}^{-1} = \begin{bmatrix} \frac{3}{16a^6} & 0 \\ 0 & \frac{3}{16a^6} \end{bmatrix}$.

---

(12) (for $n = 2$):

$$\frac{d}{dt}f_W(t) = \frac{(4a^2)^5}{\frac{256}{9}a^{12}}\Big[(6)(0) - (4a^2)(0)\Big] = 0$$

at $t = 0$

$$A_1'' = A_2'' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(13) Second derivative of $|\det A_i|$:

$$|\det A_i|'' = \det A_i\Big[p_i^2 + \operatorname{tr}\big(-(A_i^{-1}A_i')^2 + A_i^{-1}A_i''\big)\Big].$$

$$A_1^{-1}A_1' = \begin{bmatrix} 0 & -\frac{1}{2a} \\ \frac{1}{2a} & \frac{1}{2a} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -a^2 & -2a^2 \end{bmatrix} = \begin{bmatrix} \frac{a}{2} & a \\ -\frac{a}{2} & -a \end{bmatrix},$$

$$\Rightarrow \quad (A_1^{-1}A_1')^2 = \begin{bmatrix} -\frac{a^2}{4} & -\frac{a^2}{2} \\ \frac{a^2}{4} & \frac{a^2}{2} \end{bmatrix},$$

$$\Rightarrow \quad \operatorname{tr}\bigl(-(A_1^{-1}A_1')^2\bigr) = -\frac{a^2}{4} = p_1'.$$

$$A_2^{-1}A_2'' = \begin{bmatrix} -\frac{1}{2a} & -\frac{1}{2a} \\ \frac{1}{2a} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -a^2 & -a^2 \end{bmatrix} = \begin{bmatrix} \frac{a}{2} & \frac{a}{2} \\ 0 & 0 \end{bmatrix},$$

$$\Rightarrow \quad (A_2^{-1}A_2')^2 = \begin{bmatrix} \frac{a^2}{4} & \frac{a^2}{4} \\ 0 & 0 \end{bmatrix}, \quad \Rightarrow \quad \operatorname{tr}\bigl(-(A_2^{-1}A_2')^2\bigr) = -\frac{a^2}{4} = p_2'.$$

$$\Rightarrow \quad |\det A_1|'' = 4a^2\left[\left(-\frac{a}{2}\right)^2 - \frac{a^2}{4}\right] = 0, \qquad |\det A_2|'' = 4a^2\left[\left(\frac{a}{2}\right)^2 - \frac{a^2}{4}\right] = 0.$$

---

(14) $V'' = \dfrac{1}{2}\displaystyle\sum_{i=1}^{m-2} |\det A_i|'' = \frac{1}{2}(0+0) = 0.$

(15)

$$\left(\int_P \langle z, p\rangle\, dp\right)'' = \frac{1}{6}\sum_{i=1}^{m-2} (\det A_i)\Bigl[2p_i S_i' + (p_i^2 + p_i')S_i + S_i''\Bigr],$$

where S_i, S_i' computed in (8), and $S_i'' = a_i'' + b_i'' + c''$, $p_i' = \operatorname{tr}(-(A_i^{-1}A_i')^2 + A_i^{-1}A_i'')$ (computed in (13)).

$$\Rightarrow \quad S_1'' = 3\langle z, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\rangle = 0 = S_2'', \qquad p_1' = p_2' = -\frac{a^2}{4}.$$

$$\Rightarrow \quad (15) = \frac{4a^2}{6}\left[\left[2\left(\frac{-a}{2}\right)0 + \left(\left(\frac{-a}{2}\right)^2 - \frac{a^2}{4}\right)\langle z, \begin{bmatrix} a \\ a \end{bmatrix}\rangle + 0\right] + \left[2\left(\frac{a}{2}\right)a^2 z_2 + \left(\left(\frac{a}{2}\right)^2 - \frac{a^2}{4}\right)S_2 + 0\right]\right].$$

$$= \frac{2a^2}{3}\left[a^3 z_2\right] = \boxed{\frac{2a^5 z_2}{3}}.$$

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(16)

$$\left(\int_P \langle z, p^2\, dp\right)'' = \frac{1}{12}\sum_{i=1}^{m-2} (\det A_i)\bigl(Q_i(p_i^2 + p_i') + 2p_i Q_i' + Q_i''\bigr),$$

where Q_i from (7), Q_i' from (9), and $Q_i'' = (a_i''T_i + b_i''U_i + c''V_i) + (a_i'T_i' + b_i'U_i' + c'V_i')$, with

$$T_i = b_i + c + 2a_i, \quad U_i = a_i + c + 2b_i, \quad V_i = b_i + a_i + 2c.$$

Triangle 1:

$$T_1 = (b_1 + c + 2a_1) = \langle z, \begin{bmatrix} a \\ -a \end{bmatrix} + \begin{bmatrix} -a \\ a \end{bmatrix} + 2\begin{bmatrix} a \\ a \end{bmatrix} \rangle = 2a(z_1 + z_2).$$

$$U_1 = (a_1 + c + 2b_1) = \langle z, \begin{bmatrix} a \\ a \end{bmatrix} + \begin{bmatrix} -a \\ a \end{bmatrix} + 2\begin{bmatrix} a \\ -a \end{bmatrix} \rangle = \langle z, \begin{bmatrix} 2a \\ 0 \end{bmatrix} \rangle = 2az_1.$$

$$V_1 = (b_1 + a_1 + 2c) = \langle z, \begin{bmatrix} a \\ -a \end{bmatrix} + \begin{bmatrix} a \\ a \end{bmatrix} + 2\begin{bmatrix} -a \\ a \end{bmatrix} \rangle = 2az_2.$$

$$T_1' = (b_1' + c' + 2a_1') = \langle z, \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a^2 \end{bmatrix} + 2\begin{bmatrix} 0 \\ -a^2 \end{bmatrix} \rangle = -a^2 z_2.$$

$$U_1' = (a_1' + c' + 2b_1') = \langle z, \begin{bmatrix} 0 \\ -a^2 \end{bmatrix} + \begin{bmatrix} 0 \\ a^2 \end{bmatrix} \rangle = 0.$$

$$V_1' = (b_1' + a_1' + 2c') = \langle z, \begin{bmatrix} 0 \\ -a^2 \end{bmatrix} + 2\begin{bmatrix} 0 \\ a^2 \end{bmatrix} \rangle = a^2 z_2.$$

Triangle 2:

$$T_2 = (b_2 + c + 2a_2) = \langle z, \begin{bmatrix} -a \\ -a \end{bmatrix} + \begin{bmatrix} -a \\ a \end{bmatrix} + 2\begin{bmatrix} a \\ -a \end{bmatrix} \rangle = \langle z, \begin{bmatrix} 0 \\ -2a \end{bmatrix} \rangle = -2az_2.$$

$$U_2 = (a_2 + c + 2b_2) = \langle z, \begin{bmatrix} a \\ -a \end{bmatrix} + \begin{bmatrix} -a \\ a \end{bmatrix} + 2\begin{bmatrix} -a \\ -a \end{bmatrix} \rangle = \langle z, \begin{bmatrix} -2a \\ -2a \end{bmatrix} \rangle = -2a(z_1 + z_2).$$

$$V_2 = (b_2 + a_2 + 2c) = \langle z, \begin{bmatrix} -a \\ -a \end{bmatrix} + \begin{bmatrix} a \\ -a \end{bmatrix} + 2\begin{bmatrix} -a \\ a \end{bmatrix} \rangle = \langle z, \begin{bmatrix} -2a \\ 0 \end{bmatrix} \rangle = -2az_1.$$

$$T_2' = (b_2' + c' + 2a_2') = \langle z, \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rangle = a^2 z_2.$$

$$U_2' = (a_2' + c' + 2b_2') = 0 + \langle z, \begin{bmatrix} 0 \\ a^2 \end{bmatrix} \rangle + 0 = a^2 z_2.$$

$$V_2' = (b_2' + a_2' + 2c') = 2z_2 a^2.$$

$$\Rightarrow \quad Q_1'' = (0 \cdot T_1 + 0 \cdot U_1 + 0 \cdot V_1) + ((-a^2 z_2)(-a^2 z_2) + (a^2 z_2)(0) + (a^2 z_2)^2) = 2a^4 z_2^2.$$

(Detailed: $(a_1' T_1' + b_1' U_1' + c' V_1') = (-a^2 z_2)(-a^2 z_2) + 0 \cdot 0 + (-a^2 z_2)(a^2 z_2) = a^4 z_2^2 - a^4 z_2^2 + (a^2 z_2)^2 = 2a^4 z_2^2$.)

$$Q_2'' = (a_2' T_2' + b_2' U_2' + c' V_2') = 0 + 0 + (a^2 z_2)(2 z_2 a^2) = 2a^4 z_2^2.$$

$$\begin{aligned}
\Rightarrow \quad (16) = \frac{4a^2}{12} \Bigg[& Q_1 \left(\left(-\frac{a}{2} \right)^2 - \frac{a^2}{4} \right) + 2\left(-\frac{a}{2} \right)(-2a^2 z_1 z_2) + 2a^4 z_2^2 \\
& + Q_2 \left(\left(\frac{a}{2} \right)^2 - \frac{a^2}{4} \right) + 2\left(\frac{a}{2} \right)(-2a^3 z_1 z_2) + 2a^4 z_2^2 \Bigg] \\
= \frac{a^2}{3} & \left[2a^4 z_1 z_2 - 2a^4 z_1 z_2 + 4a^4 z_2^2 \right] = \boxed{\frac{4a^6 z_2^2}{3}}.
\end{aligned}$$

---

(17) Second derivative of $\bar{C}$:

$$\bar{C}'' = (0) \left(\int_P \langle z, p \rangle^2 \, dp \right) + 2(0) \left(\int_P \langle z, p \rangle^2 \, dp \right)' + (4a^2) \left(\frac{4a^6 z_2^2}{3} \right) - 2 \left[\left(-\frac{2}{3} a^4 z_1 \right)^2 + 0 \cdot \left(\int_P \langle z, p \rangle \, dp \right)'' \right]$$

$$= \frac{16a^8 z_2^2}{3} - 2 \left[\frac{4}{9} a^8 z_1^2 \right] = \frac{16a^8 z_2^2}{3} - \frac{8}{9} a^8 z_1^2.$$

$$\Rightarrow \quad \bar{C}'' = \begin{bmatrix} -\frac{8}{9} a^8 & 0 \\ 0 & \frac{16}{3} a^8 \end{bmatrix}.$$

$$\bar{C}' \bar{C}^{-1} = \begin{bmatrix} 0 & -\frac{8}{3} a^7 \\ -\frac{8}{3} a^7 & 0 \end{bmatrix} \begin{bmatrix} \frac{3}{16a^6} & 0 \\ 0 & \frac{3}{16a^6} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{a}{2} \\ -\frac{a}{2} & 0 \end{bmatrix}, \qquad (\bar{C}' \bar{C}^{-1})^2 = \begin{bmatrix} \frac{a^2}{4} & 0 \\ 0 & \frac{a^2}{4} \end{bmatrix}.$$

(18) $g = V + \mathrm{tr}[\bar{C}^{-1} C'] = 4a^2(0) = \boxed{0}$.

(19)

$$\left(\frac{g}{2}\right)' = \operatorname{tr}\Big[-(\bar{C}^{-1}\bar{C}')^2 + \bar{C}^{-1}\bar{C}''\Big] = \operatorname{tr}\left[-\begin{bmatrix} \frac{a^2}{4} & 0 \\ 0 & \frac{a^2}{4} \end{bmatrix} + \begin{bmatrix} \frac{3}{16a^6} & 0 \\ 0 & \frac{3}{16a^6} \end{bmatrix}\begin{bmatrix} -\frac{8}{9}a^8 & 0 \\ 0 & \frac{16}{3}a^8 \end{bmatrix}\right]$$

$$= -\frac{a^2}{2} + \frac{5a^2}{6} = \boxed{\frac{a^2}{3}}.$$

---

(20) (for $n = 2$):

$$f_W''(t) = \frac{(4a^2)^4}{\frac{256}{9}a^{12}}\left[(5(0) - (0))(6(0) - 0) + (6)(4a^2)(0) - (4a^2)^2\left(\frac{a^2}{3}\right) - (0)(0)\right]$$

$$= \frac{9}{a^4}\left[-\frac{16a^6}{3}\right] = \boxed{-48\,a^2}.$$

Chapter 8

Discussion & Future Work

8.1 Software

The formulas derived in this paper have been implemented in a Python package available at:

`https://github.com/jchaitheguy/isotropic_ratio`

Given a polygon specified by its vertices, the software computes the first and second derivatives of the isotropic ratio $\frac{d}{dt}f_W(t)$ and $\frac{d^2}{dt^2}f_W(t)$ as derived in Chapter 6.1 and Chapter 6.2. This provides a practical tool for running computational experiments on the landscape of the isotropic ratio over polygons, with the aim of informing conjectures toward the Strong Slicing Conjecture in higher dimensions.

Unfortunately, we did not have enough time to use this python script to run tests on the number and characteristics of the eigenvalues in the isotropic ratio's Hessian for all P. However, we were able to verify that the script's outputs do match up with the manual computations that we made in Chapter 7.

Chapter 9

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