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Strict Steiner Symmetrization for Polygons

Joseph Chai
Princeton University, jc6944@princeton.edu

Sun-Yung Alice Chang
Princeton University, syachang@math.princeton.edu

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Strict Steiner Symmetrization for Polygons

Joseph Chai
Advisor: Sun-Yung Alice Chang

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Abstract

In this note, we introduce a method called Strict Steiner Symmetrization (see Section 2.2) that is an altered form of Steiner Symmetrization that fixes the number of vertices. The context of this paper will be to prove the Isoperimetric Inequality in $\mathbb{R}^2$ via approximation by polygons. Specifically, to establish that among all n-gon domains in the plane, the regular m-gon, $m \geq n$, uniquely minimizes the isoperimetric ratio and that the limit of this regular polygon (that is the circle) achieves equality in the isoperimetric inequality. This note is an extracted portion of my junior thesis (A Polygonal Proof of the Isoperimetric Inequality in $\mathbb{R}^2$) [1] where we focus on the Strict Steiner Symmetrization Method.

Acknowledgment

I am especially grateful to my advisor, Dr. Alice Chang, and a close friend, Shouda Wang, for their patience, support, and invaluable guidance throughout this project. Their insights and encouragement were instrumental in helping me navigate moments of confusion and approach the material with greater clarity. I'd like to also thank my family and friends for sacrificing so much for me and encouraging me.

1 Introduction

The motivation for Strict Steiner Symmetrization stems from efforts to prove the isoperimetric inequality using polygons. Briefly summarizing, the isoperimetric inequality states that among all simple and closed curves with fixed perimeter M, the circle with perimeter M maximizes area. One can prove this inequality by starting with its application on polygons, as polygons approximate curves. If γ is the simple closed curve, let $L(\gamma)$ denote its length, and $A(\gamma)$ denote the area bounded by γ. Then, the isoperimetric inequality states that

$$\frac{L(\gamma)^2}{A(\gamma)} \geq 4\pi. \tag{1}$$

Let P_n denote a n-sided polygon. An analogous version of the isoperimetric inequality on polygons states that

$$\frac{L(P_n)^2}{A(P_n)} \geq \frac{L(\overline{P_m})^2}{A(\overline{P_m})}. \tag{2}$$

where $m \geq n$ and $\overline{P_m}$ is the regular m-gon (equal side lengths and angles).

We will rigorously define $\overline{P_m}$ to be the convex hull of m vertices $(v_1, v_2, .., v_m)$, where $v_i = (R\cos(\frac{2\pi i}{m}), R\sin(\frac{2\pi i}{m}))$, R is the radius of the circle which P_m is inscribed within, and center at the origin. Our polygons live within the *Hausdorff Metric Topology.*

Given two set of points $A, B \in \mathbb{R}^2$, we define the *Hausdorff Distance* between the two sets as

$$d_H(A,B) = \max\{\sup_{x\in A}\inf_{y\in B} ||x-y||, \sup_{y\in B}\inf_{x\in A} ||x-y||\}$$

We denote B_R to be the closed disk with radius R. Let, $\overline{P_m}$ denote a sequence of regular polygons with vertices with distance R from the origin. It follows, $\forall m, d_H(\overline{P_m}, B_R) = R(1-\cos(\frac{2\pi}{n}))$. Hence, $d_H(\overline{P_m}, B_R) \xrightarrow{n\to\infty} 0 \Rightarrow \lim_{m\to\infty}\overline{P_m} = B_R$

Thus, $\lim_{m\to\infty}\frac{L(\overline{P_m})^2}{A(\overline{P_m})} = 4\pi$ and we will have proved the Isoperimetric Inequality in $\mathbb{R}^2$. Please note throughout the rest of this paper, that the symmetrization methods mentioned in this paper are not sufficient to prove the isoperimetric inequality. Rather, we are simply motivating these symmetrization techniques using the isoperimetric inequality. Below, we define the assumptions and notations that we will use throughout the rest of this paper.

1.1 Assumptions & Notations

Before we delve into the details, we present some statements about the spaces we will be working with and provide justifications for them. Please refer to [1] for more background on these justifications.

1. It suffices to only consider convex polygons in this paper. Suppose, we started off with a non-convex polygon P_n. Then there is some nonconvex region between some set of vertices v_a, v_b, v_c where $a, b, c \in [1, n]$. Then, we can simply reflect v_b to the other side of the line segment $\overline{v_a v_c}$. Denote this new polygon as P_n'. The reflection has preserved perimeter but increased area. Hence $IR(P_n') \leq IR(P_n)$ and all convex polygons possess smaller isoperimetric ratios than nonconvex polygons.

2. We will only provide a proof of Strict Steiner Symmetrization in $\mathbb{R}^2$.

3. Let P_n be some n-sided polygon. Then, $IR(P_n)$ denotes $\frac{L^2(P_n)}{A(P_n)}$ (i.e., the isoperimetric ratio of P_n).

4. Let II be an abbreviation for the Isoperimetric Inequality, SS for Steiner Symmetrization, and SSS for Strict Steiner Symmetrization.

5. Let $S_n^M = \{P_a | a \leq n,\ L(P_a) = M\}$ (i.e., the set of polygons with at most n sides with perimeter M).

6. Let $\overline{P_n}$ denote the regular n-gon $\in S_n^M$ with perimeter M.

7. Let P_n be a general n-gon in S_n^M

8. In this paper, we will not prove that when $n = 3$, the equilateral triangle minimizes IR among all $P_3 \in S_3^M$, and likewise when $n = 4$, the square minimizes IR. (You can refer to Section 0.2 and 0.3 in [1], and [6] and [7] for the proofs)

9. The orientation of labeling vertices will be clockwise.

10. Let $a, b \in \mathbb{R}$, then $\mathbb{N}_{[a,b]}$ denotes the set of integers between a and b inclusive. A similar result holds for $\mathbb{Z}_{[a,b]}$.

1.2 Different Approaches to Proving the II on Polygons

There are generally two different approaches to show (2):

1. Direct Proof: Define an iterative process $\theta^k : S_n^M \to S_m^{M'}$ where $m \geq n,\ M' \leq M$ such that for all $P_n \in S_n^M,\ L(\theta^k(P_n)) \leq L(P_n),\ A(\theta^k(P_n)) \geq A(P_n)$ and $\lim_{k\to\infty} \theta^k(P_n) = \overline{P_m}$.

2. Indirect Proof: Show the existence of a polygon $P^* \in S_n^M$ that minimizes the isoperimetric ratio, among all $P \in S_n^M$. Show that if P^* exists, it must be a regular polygon.

We will focus only on direct proofs in this paper.

1.3 An Direct Proof Attempt of the Isoperimetric Inequality on Polygons: Steiner Symmetrization (SS)

Setup: Let P_n be a polygon with n sides and perimeter M. Here, we use Assumption & Notations (9) (all vertices are labeled clockwise). Denote $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ as the set of vertices of P_n. Choose any two consecutive vertices, v_i and $v_{(i+1) \bmod n}$, and let $\bar{l}$ denote the edge $v_i v_{(i+1) \bmod n}$. Let β denote the midpoint of $\bar{l}$. Draw a line E such that $E \perp \bar{l}$ and E goes through β.

Step 1: Splitting
For all $v_h \in P_n$ where $h \notin \{i, (i+1) \bmod n\}$, draw a line $\overleftrightarrow{l_h}$ such that $\overleftrightarrow{l_h} || \bar{l}$ and passes through v_h (illustrated below in Figure 1).

The polygon P_n is now partitioned into triangles and trapezoids. Let $S = \{S_1, S_2, \ldots, S_k\}$ denote the parts of this partition, labeled from bottom to top (with the bottom part containing edge $\bar{l}$).

Step 2: Piecewise Symmetrization

Steiner proposed reducing $IR(P_n)$ by reducing $IR(S_i)$ for all $S_i \in S$. We apply the following casework to each part $S_i \in S$ starting from S_1:

1. If S_i is a triangle, we fix its base and slide its top vertex along a line parallel to the base to a point where the triangle becomes isosceles.

2. If S_i is a trapezoid with bases b_1, b_2 (top and bottom bases respectively) and height h, we simply fix b_1 and shift b_2 along its respective $\overleftrightarrow{l_h}$ such that E bisects both b_1 and b_2. (It is left as an exercise to the reader to show that such an adjustment fixes the area but reduces the leg lengths of the trapezoid, thereby reducing the perimeter of the entire trapezoid).

Notice, in both cases, we are always fixing the area of each part and reducing the length of its shared edges with the entire polygon. Hence, under both cases, we fix the area of the entire polygon and guarantee non-increasing perimeter.

After processing all parts, we obtain one iteration of Steiner Symmetrization. We denote P_n^{j*} as the resulting polygon after j iterations of Steiner Symmetrization. One completes another iteration of Steiner Symmetrization by fixing a new edge $\overline{l'}$ of P_n^1* and repeating the same algorithm above.

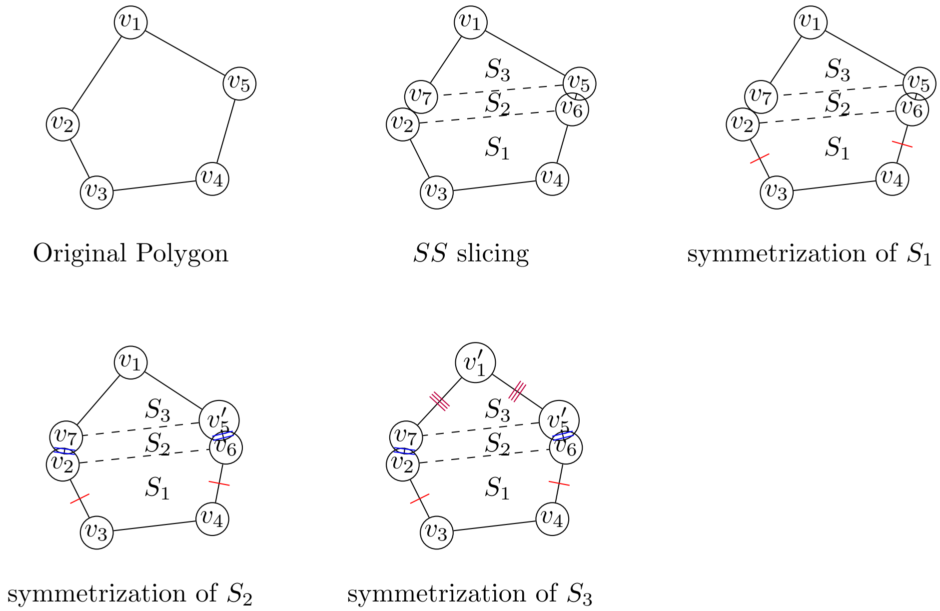

Figure 1: One iteration of SS on a polygon on edge v_3v_4

1.3.1 When does SS converge?

Consider any polygon P_n with $L(P_n) = M$. Let $V = \{v_0, v_1, \ldots, v_{n-1}\}$ denote the set of vertices of P_n, labeled clockwise.

For all $i \in \mathbb{Z}_{[0,n-1]}$, let $A = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ denote the set of angles of P_n, where α_i is the angle at vertex v_i.

Let $E = \{e_0, e_1, \ldots, e_{n-1}\}$ denote the set of edges of P_n, where e_i is the edge connecting v_i and $v_{(i+1) \bmod n}$.

One can no longer apply Steiner Symmetrization on some polygon P_n when P_n is symmetric with respect to every edge (i.e., For all edges of P_n you can reflect P_n along the edge's perpendicular bisector and no asymmetry exists within the reflected polygon). Unfortunately, this assumption does not imply that P_n is regular and Steiner was unable to prove the existence of the circle in being the minimizer of the isoperimetric ratio in $\mathbb{R}^2$. A few other folks such as Weierstrass and Schwarz later proved the isoperimetric inequality. A multitude of proofs to the isoperimetric inequality can be found in [2]

1.3.2 Why do we still need Strict Steiner Symmetrization?

As seen in Figure 1, SS preserves area in both the trapezoid and triangle case but reduces the perimeter of each piece while fixing the bases. Hence, we have already found an iterative method that preserves area and minimizes perimeter, thereby reducing the isoperimetric ratio of our polygon. However, we ask whether we can show something stronger: given any polygon $P_n \in S_n^M$, there exists an iterative process that reduces isoperimetric ratio and fixes the number of vertices? The main result of this paper is precisely the theorem below.

Theorem 1.1. *Given any polygon $P_n \in S_n^M$, among all n-sided polygons in S_n^M, $IR(\overline{P_n}) \leq IR(P_n)$*

Proof. The proof will be provided in Section 2 via the Strict Steiner Symmetrization iterative method. □

2 Main Results

2.1 Strict Steiner Symmetrization (SSS)

2.1.1 Setup

Let P_n be a polygon with n sides and perimeter M. Denote $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ as the set of vertices of P_n. Choose any two consecutive vertices, v_i and $v_{(i+1) \bmod n}$, and let $\bar{l}$ denote the edge $v_i v_{(i+1) \mod n}$. Let β denote the midpoint of $\bar{l}$. Draw a line E such that $E \perp \bar{l}$ and E goes through β.

Step 1: Splitting

Instead of slicing according to SS, we can decompose P_n into triangles and quadrilaterals that share bases. Specifically:

- If n is odd, we define two sets of vertices $A = \{a_1 = 2, a_2 = 3, .., a_k = \frac{n-1}{2}\}, B = \{b_1 = n, b_2 = n-1, b_3 = n-2, .., b_k = \frac{n-1}{2} + 1\}$, and $\forall i, 1 \leq i \leq k$ draw a dashed line between vertices a_i and b_i.

- If n is even, we define two sets of vertices $A = \{a_1 = 2, a_2 = 3, .., a_k = \frac{n}{2}\}, B = \{b_1 = n, b_2 = n-1, b_3 = n-2, .., b_k = \frac{n}{2} + 2\}$, and $\forall i, 1 \leq i \leq k$ draw a dashed line between vertices a_i and b_i.

At this point, P_n is now partitioned into adjacent triangles and quadrilaterals. Let $S = \{S_1, S_2, \ldots, S_k\}$ denote the parts of this partition, labeled from top to bottom.

Step 2: New Piecewise Symmetrization

1. If S_i is a triangle, apply the isosceles transformation described in Section 1.3.1 (same as in SS). (Fix area, minimize perimeter)
2. If S_i is a quadrilateral $ABCD$, fix both position and length of $\overline{AB}$, fix just the length of $\overline{CD}$ and adjust vertices C and D to vertices C' & D' respectively such that $\overline{BC'} = \overline{AD'}$, and $\overline{BC} + \overline{AD} = 2\overline{BC'}$. We provide a proof in Section 2.2.2 below to show that such a case-optimization does indeed maximize area under fixing perimeter for general quadrilaterals. (Fix perimeter, maximize area)

Unlike in SS, where we only fix area while minimizing perimeter, here we have a hybrid of both.

2.1.2 Example: SSS on 5-gon

Proof. Let P_1 denote our 5-gon with vertices $\{v_1, v_2, v_3, v_4, v_5\}$ labeled clockwise, with constants a, b where $\overline{v_2v_5} = a, \overline{v_3v_4} = b$. After splitting as described above, let S_1 denote the $v_1v_2v_3$ triangle, and S_2 denote the $v_2v_3v_4v_5$ quadrilateral, where $P_1 = S_1 \cup S_2$. Then under our first step of SSS, $S_1 \to S_1'$ where S_1' is the isosceles triangle $v_1'v_2v_3$ where v_1 has been adjusted to v_1'. Let $P_2 = S_1' \cup S_2$. As shown above in previous sections $L(S_1) \geq L(S_1'), A(S_1) = A(S_1') \Rightarrow L(P_1) \geq L(P_2), A(P_1) = A(P_2)$.

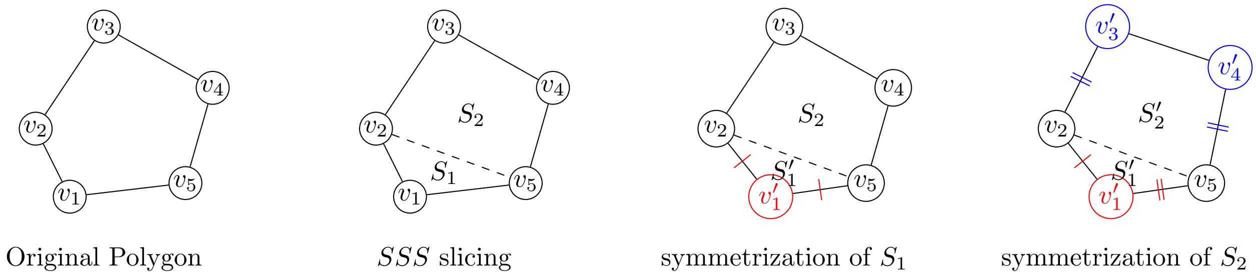

Figure 2: One iteration of SSS on a pentagon

After symmetrizing the bottom triangle, we now symmetrize the upper quadrilateral, S_2 (by simply moving $v_4 \to v_4', v_3 \to v_3'$, fixing v_2, v_5).

Intuition: One can think of having a closed string of length $L(S_2)$ that outlines S_2. Then, one slides two pipes p_1, p_2 of respective fixed lengths a, b. Next, fix one pipe, say p_1 and move the other pipe, p_2, below p_1 such that $p_1 || p_2$ and their midpoints are perfectly aligned on top of each other. We let S_2' denote the resultant quadrilateral.

Let S_2' denote the symmetrized S_2 with $L(S_2) = L(S_2')$. We borrow Bretschneider's Formula [5] for the area of a general quadrilateral. Let $\theta_1 = \angle v_3 v_4 v_5, \theta_2 = \angle v_5 v_2 v_3$. Let $h = \overline{v_4 v_5}, k = \overline{v_2 v_3}, s = \frac{a+b+h+k}{2}$ (i.e. semiperimeter), $\theta = \theta_1 + \theta_2$. Then,

$$A(S_2) = \sqrt{(s-a)(s-b)(s-h)(s-k) - abhk\cos^2(\frac{\theta}{2})}$$

$$\text{Let } h' = \overline{v_4' v_5}, k' = \overline{v_2 v_3'}$$

$$A(S_2') = \sqrt{(s-a)(s-b)(s-h')^2 - ab(h'^2)\cos^2(\frac{\pi}{2})} = \sqrt{(s-a)(s-b)(s-h')^2}$$

To show that the area of our quadrilateral is non-decreasing under such a quadrilateral symmetrization, we want to show that $\frac{A(S_2')}{A(S_2)} \geq 1$. Then, it suffices to show $(s-h')^2 \geq (s-h)(s-k)$

$$\text{Let } A \ = (s-h')^2 = s^2 - 2s(\frac{h+k}{2}) + (\frac{h+k}{2})^2$$

$$\text{Let } B \ = (s-h)(s-k) = s^2 - (h+k)s + hk$$

$$A - B = (\frac{h+k}{2})^2 - hk = \frac{h^2+2hk+k^2}{4} - hk = \frac{h^2+k^2}{4} - \frac{hk}{2} = \frac{h^2+k^2-2hk}{4} = \frac{(h-k)^2}{4}$$

$$(h-k)^2 \geq 0 \Rightarrow A - B \geq 0$$

Let $P_3 = S_1' \cup S_2'$, then $A(P_2) \leq A(P_3), L(P_2) = L(P_3) \Rightarrow \frac{L^2(P_1)}{A(P_1)} \geq \frac{L^2(P_2)}{A(P_2)} \geq \frac{L^2(P_3)}{A(P_3)}$. Such reasoning can be applied inductively on higher n-gons and we provide the inductive proof below. $\square$

2.1.3 Showing SSS indeed reduces IR

Lemma 2.0.1. *$\forall P_n \in S_n^M \setminus \bigcup_{i=3}^{n-1} S_i^M$, Let $P_n' = SSS(P_n)$, then $L(P_n) \geq L(P_n'), A(P_n) \leq A(P_n')$.*

Proof. We prove the result by induction on n.

Base Case: $n = 5$.
Shown in Section 2.1.2

Inductive Hypothesis:
Assume that Lemma 2.1 holds for some arbitrary $n \geq 5$.

Inductive Step:
We must show that Lemma 2.1 is also true for $(n+1)$-gons, under this assumption.

Let P_{n+1} be some convex polygon with $n+1$ vertices and perimeter M.

Let $V = \{v_1, v_2, ..., v_{n+1}\}$ denote the vertices of P_{n+1} labeled clockwise.

1. We can partition P_{n+1} using the strategy mentioned in Section 2.1.1 Step (1).

2. Notice, in our partitioning strategy in part 1a), we always end up with at least one triangle T in the partition. Hence, let P_n be the polygon $P_{n+1} \setminus T$. Let a denote the edge that separates T and P_n. Note: a is also the edge that separates T' and P_n' as it is preserved under SSS.

3. Let $P_{n+1}' = SSS(P_{n+1}),\ T' = SSS(T)$.

   Recall we want to show that $L(P_{n+1}) \geq L(P_{n+1}'), A(P_{n+1}) \leq A(P_{n+1}')$.

   We know that $L(T) \geq L(T')$ by Section 0.2.1 in [1]

$$L(P_{n+1}) = L(P_n) + L(T) - 2|a|$$

$$L(P_{n+1}') = L(P_n') + L(T') - 2|a|$$

   By our Induction Hypothesis, $L(P_n') \leq L(P_n)$ and we have shown that $L(P_{n+1}) \geq L(P_{n+1}')$.

   Next,

$$\begin{aligned} A(P_{n+1}) &= A(P_n) + A(T) \\ &\leq A(P_n') + A(T') = A(P_{n+1}') \end{aligned}$$

   Since both the base case and the inductive step have been established, Lemma 2.1. holds for all $n \geq n_0$ by mathematical induction.

□

Corollary 2.0.1. *$\forall n \in \mathbb{N}, n \geq 5$, Let $P_n' = SSS(P_n), IR(P_n) \geq IR(P_n')$*

Proof.

$$IR(P_{n+1}) = \frac{L^2(P_{n+1})}{A(P_{n+1})} = \frac{(L(P_n) + L(T) - 2|a|)^2}{A(P_n) + A(T)}$$

$$IR(P'_{n+1}) = \frac{L^2(P'_{n+1})}{A(P'_{n+1})} = \frac{(L(P'_n) + L(T') - 2|a|)^2}{A(P'_n) + A(T')}$$

We know

$$L(T) \geq L(T') \tag{3}$$

$$A(T) = A(T') \tag{4}$$

$$L(P_n) \geq L(P'_n) \tag{5}$$

$$A(P_n) \leq A(P'_n) \tag{6}$$

(1) & (2) is known from Section 0.2.1 in [1]. (3) & (4) is known from Lemma 2.0.1. It is clear that $A(P'_{n+1}) \geq A(P_{n+1})$. Hence, what is left to show is $L^2(P'_{n+1}) \leq L^2(P_{n+1})$. You can expand out the fraction into their individual terms to see that $IR(P_{n+1}) \geq IR(P'_{n+1})$. A simpler way of proving this inequality is to notice that $L(P'_n), L(P_n) \geq |a|$ by the triangle inequality. Hence, $L(P'_n) + L(T') \leq L(P_n) + L(T)$ and we are done.

□

3 Conclusion

To conclude, our initial problem was to find a simple and understandable proof to the classic Isoperimetric Inequality in $\mathbb{R}^2$. We know that polygons can approximate curves, so if we can prove II on convex polygons we can get really close to proving it on simple and closed curves. Please refer back to the Introduction section for the details of how this construction is actually made. On the journey of proving the isoperimetric inequality on polygons, we realized that people in the past have used generally two types of approaches: indirect and direct proofs. Although, we do not prove the isoperimetric inequality we encountered Steiner Symmetrization (a process that reduces IR but does not prove existence of the circle as the minimizer to IR). Nonetheless, we noticed that the resultant polygon of SS can have more vertices than the polygon you started off with and we wondered whether SS could be re-engineered to fix the number of vertices. Indeed, (Strict Steiner Symmetrization) does the job.

3.0.1 Applications

There aren't many big proofs or complicated machinery we work with here, but SSS is worth sharing! One of the potential applications of SSS is in solving the Polya conjecture on the first eigenvalue of Dirichlet Laplacian for regular N-gons. It is known that for $n = 3, n = 4$, the equilateral triangle and square give the minimal first eigenvalue among all n-gons with fixed area. However, the proof for $n \geq 5$ cases with Steiner Symmetrization fails due to the increase of vertices. More details can be found in [8, 9].

3.1 Limitations and Future Work

1. We do not include any work detailing Strict Steiner Symmetrization applied to higher dimensions.

2. Strict Steiner Symmetrization may not preserve convexity.

3. The symmetrization techniques mentioned in this paper do not prove the isoperimetric inequality, they are simply motivated by it.

4. We do not include any in-depth analysis on the computational complexity of both methods. (Some basic analysis on the average convergence rate in iterations can be found in the codebase)

4 References

4.1 Codebase

For those who might be more interested in the computational proof feel free to check out the codebase written in Python.

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Joseph Chai, **Sun-Yung Alice Chang (Project Advisor)**
Princeton University
Email: `jc6944@princeton.edu`, `syachang@math.princeton.edu`